MEM6804 Modeling and Simulation for Logistics & Supply Chain
物流与供应链建模与仿真

Theory Analysis

Lecture 3: Queueing Models

SHEN Haihui 沈海辉

Sino-US Global Logistics Institute
Shanghai Jiao Tong University

shenhaihui.github.io/teaching/mem6804f
shenhaihui@sjtu.edu.cn

Spring 2021 (full-time)

上海交通大學 SHANGHAI JIAO TONG UNIVERSITY
董浩云航运与物流研究院
CY TUNG Institute of Maritime and Logistics
中美物流研究院（工程系统管理研究院）
Sino-US Global Logistics Institute (Institute of Industrial & System Engineering)

Contents

- Queues (or waiting lines) are EVERYWHERE!

- Queues (or waiting lines) are EVERYWHERE!
- Queues are an unavoidable component of modern life.
  - E.g., in hospital, stores, bank, call center (online service), etc.

Figure: Queues in Hospital

Figure: Queues in Store (*from* *The Sun*)

Figure: Queues in Bank

Figure: Queues in Bank (No requirement to *stand physically* in queues)

Figure: Queue in Online Service

- Queues (or waiting lines) are EVERYWHERE!
- Queues are an unavoidable component of modern life.
    - E.g., in hospital, stores, bank, call center (online service), etc.

- Queues (or waiting lines) are EVERYWHERE!
- Queues are an unavoidable component of modern life.
  - E.g., in hospital, stores, bank, call center (online service), etc.
  - Although we don't like standing in a queue, we appreciate the fairness that it imposes.

- Queues (or waiting lines) are EVERYWHERE!
- Queues are an unavoidable component of modern life.
  - E.g., in hospital, stores, bank, call center (online service), etc.
  - Although we don't like standing in a queue, we appreciate the fairness that it imposes.
- Queues are not just for humans, however.
  - E.g., email system, printer, manufacturing line, etc.

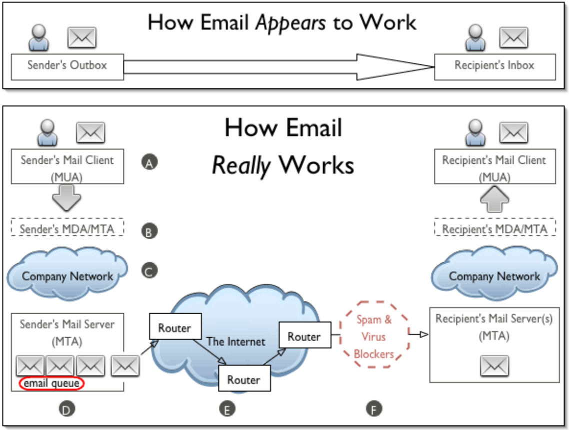


Figure: Queue in Mail Server (*from* *OASIS*)

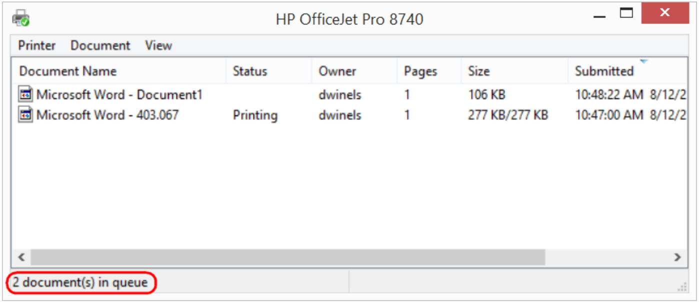


Figure: Queue in Printer

Figure: Queues (Inventories) in Manufacturing Line (*from* *Estes*)

- Queues (or waiting lines) are EVERYWHERE!
- Queues are an unavoidable component of modern life.
  - E.g., in hospital, stores, bank, call center (online service), etc.
  - Although we don't like standing in a queue, we appreciate the fairness that it imposes.
- Queues are not just for humans, however.
  - E.g., email system, printer, manufacturing line, etc.

- Queues (or waiting lines) are EVERYWHERE!
- Queues are an unavoidable component of modern life.
  - E.g., in hospital, stores, bank, call center (online service), etc.
  - Although we don't like standing in a queue, we appreciate the fairness that it imposes.
- Queues are not just for humans, however.
  - E.g., email system, printer, manufacturing line, etc.
  - Manufacturing systems maintain queues (called inventories) of raw materials, partly finished goods, and finished goods via the manufacturing process.

- Queues (or waiting lines) are EVERYWHERE!
- Queues are an unavoidable component of modern life.
  - E.g., in hospital, stores, bank, call center (online service), etc.
  - Although we don't like standing in a queue, we appreciate the fairness that it imposes.
- Queues are not just for humans, however.
  - E.g., email system, printer, manufacturing line, etc.
  - Manufacturing systems maintain queues (called inventories) of raw materials, partly finished goods, and finished goods via the manufacturing process.
- Queues in logistics management:
  - The unloading and loading process at warehouse;
  - The process of sorting out goods according to the orders;
  - Customers go to the station to pick up packages, etc.

- Typically, a queueing system consists of a stream of "**customers**" (humans, goods, messages) that
  - arrive at a service facility;
  - wait in the **queue** according to certain discipline;
  - get served;
  - finally depart.

- Typically, a queueing system consists of a stream of "**customers**" (humans, goods, messages) that
  - arrive at a service facility;
  - wait in the **queue** according to certain discipline;
  - get served;
  - finally depart.
- A lot of real-world systems can be viewed as queueing systems, e.g.,
  - service facilities
  - production systems
  - repair and maintenance facilities
  - communications and computer systems
  - transport and material-handling systems, etc.

- Typically, a queueing system consists of a stream of "**customers**" (humans, goods, messages) that
  - arrive at a service facility;
  - wait in the **queue** according to certain discipline;
  - get served;
  - finally depart.
- A lot of real-world systems can be viewed as queueing systems, e.g.,
  - service facilities
  - production systems
  - repair and maintenance facilities
  - communications and computer systems
  - transport and material-handling systems, etc.
- Queueing models are mathematical representation of queueing systems.

- Queueing models may be
  - *analytically solved using queueing theory* when they are simple (highly simplified); or
  - *analyzed through simulation* when they are complex (more realistic).

- Queueing models may be
  - *analytically solved using queueing theory* when they are simple (highly simplified); or
  - *analyzed through simulation* when they are complex (more realistic).

- Studied in either way, queueing models provide us a powerful tool for designing and evaluating the performance of queueing systems.

- Queueing models may be
  - *analytically solved using queueing theory* when they are simple (highly simplified); or
  - *analyzed through simulation* when they are complex (more realistic).

- Studied in either way, queueing models provide us a powerful tool for designing and evaluating the performance of queueing systems.

- They help us do this by answering the following questions (and many others):
  1. How many customers are there in the queue (or station) on average?
  2. How long does a typical customer spend in the queue (or station) on average?
  3. How busy are the servers on average?

- *Simple queueing models solved analytically*:
  - Get rough estimates of system performance with negligible time and expense.
  - *More importantly, understand the dynamic behavior of the queueing systems and the relationships between various performance measures.*
  - Provide a way to verify that the simulation model has been programmed correctly.

- *Simple queueing models solved analytically*:
  - Get rough estimates of system performance with negligible time and expense.
  - *More importantly, understand the dynamic behavior of the queueing systems and the relationships between various performance measures.*
  - Provide a way to verify that the simulation model has been programmed correctly.
- *Complex queueing models analyzed through simulation*:
  - Allow us to incorporate arbitrarily fine details of the system into the model.
  - Estimate virtually any performance measure of interest with high accuracy.

- *Simple queueing models solved analytically*:
  - Get rough estimates of system performance with negligible time and expense.
  - *More importantly, understand the dynamic behavior of the queueing systems and the relationships between various performance measures.*
  - Provide a way to verify that the simulation model has been programmed correctly.
- *Complex queueing models analyzed through simulation*:
  - Allow us to incorporate arbitrarily fine details of the system into the model.
  - Estimate virtually any performance measure of interest with high accuracy.
- This lecture focuses on the classical analytically solvable queueing models.

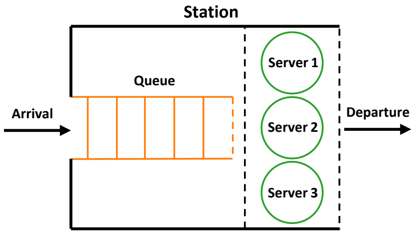
- The key elements of a queueing system are the **customers** and **servers**.
  - The term customer can refer to anything that arrives and requires service.
  - The term server can refer to any resource that provides the requested service.

- The key elements of a queueing system are the **customers** and **servers**.
  - The term customer can refer to anything that arrives and requires service.
  - The term server can refer to any resource that provides the requested service.
- The term **station** means the entire or part of the system, which contains all the identical servers and the queue.

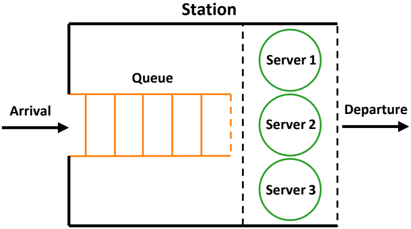
- The key elements of a queueing system are the **customers** and **servers**.
  - The term customer can refer to anything that arrives and requires service.
  - The term server can refer to any resource that provides the requested service.
- The term **station** means the entire or part of the system, which contains all the identical servers and the queue.
- Suppose that there is only **one queue** in one station.

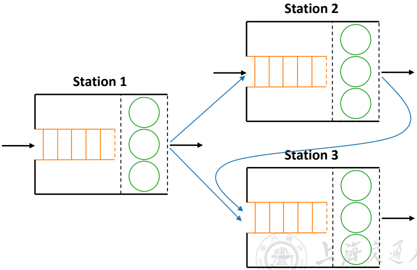
- The key elements of a queueing system are the **customers** and **servers**.
  - The term customer can refer to anything that arrives and requires service.
  - The term server can refer to any resource that provides the requested service.
- The term **station** means the entire or part of the system, which contains all the identical servers and the queue.
- Suppose that there is only **one queue** in one station.
- **Capacity** is the maximal number of customers allowed in the station.
  - Number waiting in queue + number having service.
  - Finite or infinite.

- Single-station queueing system.
  - Customers simply leave after service.
  - E.g., customers arrive to buy coffee and then leave.



- Single-station queueing system.
  - Customers simply leave after service.
  - E.g., customers arrive to buy coffee and then leave.
- Multiple-station queueing system (queueing network).
  - Customers can move from one station to another (for different service), before leaving the system.
  - E.g., patients wait and get service at several different units inside a hospital.





- The **arrival process** describes how the customers come.
  - Arrivals may occur at *scheduled* times or *random* times.
  - When at random times, the **interarrival times** are usually characterized by a probability distribution.
  - Customers may arrive one at a time or in batch (with constant or random batch size).
  - Different types of customers.

- The **arrival process** describes how the customers come.
  - Arrivals may occur at *scheduled* times or *random* times.
  - When at random times, the **interarrival times** are usually characterized by a probability distribution.
  - Customers may arrive one at a time or in batch (with constant or random batch size).
  - Different types of customers.

- An customer arriving at a station:
  - if the station capacity is full:
    - the external arrival will leave immediately (called **lost**);
    - the internal arrival may wait in the previous station (may **block** the previous server).
  - if the station capacity is not full, enter the station:
    - if there is idle server in the station, get service immediately;
    - if all servers are busy, wait in the **queue**.

- Queue discipline: Which customer to serve first.
  - First-in-first-out (FIFO), or first-come-first-served (FCFS).
  - Last-in-first-out (LIFO), or last-come-first-served (LCFS).
  - Shortest processing time first.
  - Service according to priority (more than one customer types).

- Queue discipline: Which customer to serve first.
  - First-in-first-out (FIFO), or first-come-first-served (FCFS).
  - Last-in-first-out (LIFO), or last-come-first-served (LCFS).
  - Shortest processing time first.
  - Service according to priority (more than one customer types).
- Queue behavior: Actions of customers while waiting.
  - Balk: leave when they see that the line is too long.
  - Renege: leave after being in the line when they see that the line is moving too slowly.

- Queue discipline: Which customer to serve first.
  - First-in-first-out (FIFO), or first-come-first-served (FCFS).
  - Last-in-first-out (LIFO), or last-come-first-served (LCFS).
  - Shortest processing time first.
  - Service according to priority (more than one customer types).

- Queue behavior: Actions of customers while waiting.
  - Balk: leave when they see that the line is too long.
  - Renege: leave after being in the line when they see that the line is moving too slowly.

- **Service time** is the duration of service in a server.
  - *Constant* or *random* duration.
  - May depend on the customer type.
  - May depend on the time of day or the queue length.

- When without specification, the queueing models considered in this lecture shall satisfy the following:
  1. One customer type.
  2. Random arrivals (i.e., random interarrival times, iid.).
  3. No batch (or say, batch size is 1).†
  4. One queue in one station.
  5. First-come-first-served (FCFS).
  6. No balk, no renege.
  7. Random service time (depends on nothing else), iid.

† 1+2+3 ⇒ The arrival process is a *renewal process*.

- When without specification, the queueing models considered in this lecture shall satisfy the following:
  1. One customer type.
  2. Random arrivals (i.e., random interarrival times, iid.).
  3. No batch (or say, batch size is 1).†
  4. One queue in one station.
  5. First-come-first-served (FCFS).
  6. No balk, no renege.
  7. Random service time (depends on nothing else), iid.

- Even so, it is not that easy to analyze the queueing models!

† 1+2+3 ⇒ The arrival process is a *renewal process*.

- Canonical notational system proposed by Kendall (1953): $X/Y/s/K$.

- Canonical notational system proposed by Kendall (1953): $X/Y/s/K$.
  - X represents the interarrival-time distribution.
    - M: Memoryless, i.e., exponential interarrival times;
    - G: General;
    - D: Deterministic.

- Canonical notational system proposed by Kendall (1953): $X/Y/s/K$.
  - X represents the interarrival-time distribution.
    - M: Memoryless, i.e., exponential interarrival times;
    - G: General;
    - D: Deterministic.
  - Y represents the service-time distribution.
    - Same letters as the interarrival times.

- Canonical notational system proposed by Kendall (1953): $X/Y/s/K$.
  - X represents the interarrival-time distribution.
    - M: Memoryless, i.e., exponential interarrival times;
    - G: General;
    - D: Deterministic.
  - Y represents the service-time distribution.
    - Same letters as the interarrival times.
  - s represents the number of parallel servers.
    - Finite value.
    - For infinite number of servers, s is replaced by ∞.

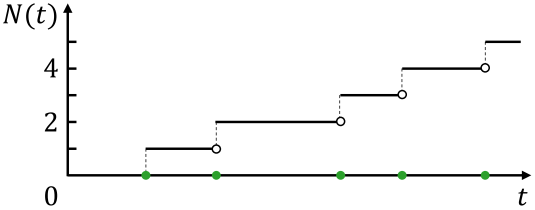
- Canonical notational system proposed by Kendall (1953): $X/Y/s/K$.
  - X represents the interarrival-time distribution.
    - M: Memoryless, i.e., exponential interarrival times;
    - G: General;
    - D: Deterministic.
  - Y represents the service-time distribution.
    - Same letters as the interarrival times.
  - s represents the number of parallel servers.
    - Finite value.
    - For infinite number of servers, s is replaced by ∞.
  - K represents the station capacity.
    - Finite value.
    - For infinite capacity, K is replaced by ∞, or simply omitted.

- Canonical notational system proposed by Kendall (1953): $X/Y/s/K$.
  - X represents the interarrival-time distribution.
    - M: Memoryless, i.e., exponential interarrival times;
    - G: General;
    - D: Deterministic.
  - Y represents the service-time distribution.
    - Same letters as the interarrival times.
  - s represents the number of parallel servers.
    - Finite value.
    - For infinite number of servers, s is replaced by ∞.
  - K represents the station capacity.
    - Finite value.
    - For infinite capacity, K is replaced by ∞, or simply omitted.
- Examples: $M/M/1$, $M/G/1$, $M/M/s/K$.

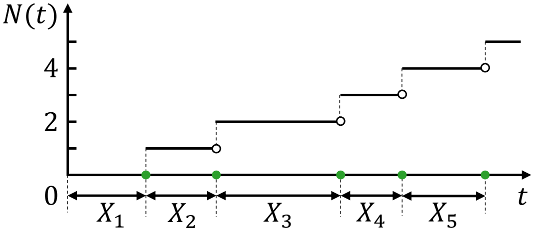
上海交通大學 SHANGHAI JIAO TONG UNIVERSITY

- A stochastic process $\{N(t),\ t \geq 0\}$ is said to be a *counting process* if $N(t)$ represents the total number of arrivals that have occurred up to time t.

- A stochastic process $\{N(t),\ t \geq 0\}$ is said to be a *counting process* if $N(t)$ represents the total number of arrivals that have occurred up to time t.



- A stochastic process $\{N(t),\ t \geq 0\}$ is said to be a *counting process* if $N(t)$ represents the total number of arrivals that have occurred up to time t.



- Let $\{X_n,\ n \geq 1\}$ denote the *interarrival times*:
  - X_1 denotes the time of the first arrival;
  - For $n \geq 2$, X_n denotes the time between the $(n-1)$st and the nth arrivals.

- **Definition 1.** The counting process $\{N(t),\ t \geq 0\}$ is called a **Poisson process** with rate λ, $\lambda > 0$, if:
  - $N(0) = 0$;
  - The process has **independent** and **stationary** increments;
  - For $t > 0$, $N(t) \sim \text{Pois}(\lambda t)$, i.e.,

$$\mathbb{P}(N(t) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots.$$

Poisson process ⊂ renewal process ⊂ counting process.

- **Definition 1.** The counting process $\{N(t),\ t \geq 0\}$ is called a **Poisson process** with rate λ, $\lambda > 0$, if:
  - $N(0) = 0$;
  - The process has **independent** and **stationary** increments;
  - For $t > 0$, $N(t) \sim \text{Pois}(\lambda t)$, i.e.,

$$\mathbb{P}(N(t) = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!},\ \ n = 0, 1, 2, \ldots.$$

- Independent Increments: The numbers of arrivals in disjoint time intervals are independent.

Poisson process $\subset$ renewal process $\subset$ counting process.

- **Definition 1**. The counting process $\{N(t),\ t \geq 0\}$ is called a **Poisson process** with rate λ, $\lambda > 0$, if:
  - $N(0) = 0$;
  - The process has **independent** and **stationary** increments;
  - For $t > 0$, $N(t) \sim \text{Pois}(\lambda t)$, i.e.,

$$\mathbb{P}(N(t) = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!},\quad n = 0, 1, 2, \ldots.$$

- Independent Increments: The numbers of arrivals in disjoint time intervals are independent.
- Stationary Increments: The distribution of number of arrivals in any time interval depends only on the length of time interval, i.e., for $s < t$, the distribution of $N(t) - N(s)$ depends only on $t - s$.

Poisson process $\subset$ renewal process $\subset$ counting process.

- **Definition 2.** The counting process $\{N(t),\ t \geq 0\}$ is called a **Poisson process** with rate λ, $\lambda > 0$, if:
  - $N(0) = 0$;
  - The process has independent and stationary increments;
  - $\mathbb{P}(N(t) = 1) = \lambda t + o(t)$;
  - $\mathbb{P}(N(t) \geq 2) = o(t)$.

- **Definition 2**. The counting process $\{N(t),\ t \geq 0\}$ is called a **Poisson process** with rate λ, $\lambda > 0$, if:
  - $N(0) = 0$;
  - The process has independent and stationary increments;
  - $\mathbb{P}(N(t) = 1) = \lambda t + o(t)$;
  - $\mathbb{P}(N(t) \geq 2) = o(t)$.

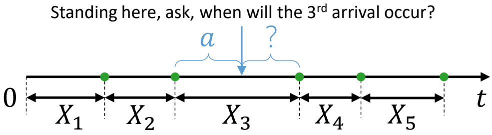
- **Definition 3**. The counting process $\{N(t),\ t \geq 0\}$ is called a **Poisson process** with rate λ, $\lambda > 0$, if:
  - $N(0) = 0$;
  - $\{X_n,\ n \geq 1\}$ is a sequence of iid random variables, and $X_n \sim \text{Exp}(\lambda)$.

- **Definition 2**. The counting process $\{N(t),\ t \geq 0\}$ is called a **Poisson process** with rate λ, $\lambda > 0$, if:
  - $N(0) = 0$;
  - The process has independent and stationary increments;
  - $\mathbb{P}(N(t) = 1) = \lambda t + o(t)$;
  - $\mathbb{P}(N(t) \geq 2) = o(t)$.

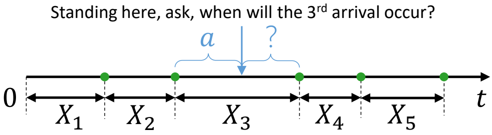
- **Definition 3**. The counting process $\{N(t),\ t \geq 0\}$ is called a **Poisson process** with rate λ, $\lambda > 0$, if:
  - $N(0) = 0$;
  - $\{X_n,\ n \geq 1\}$ is a sequence of iid random variables, and $X_n \sim \mathrm{Exp}(\lambda)$.

- **Definition 1**, **Definition 2** and **Definition 3** are equivalent.

- **Question 1**: When will the next appear?

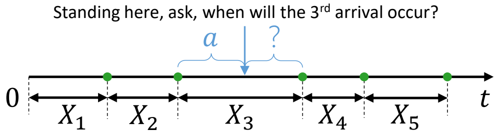


- **Question 1**: When will the next appear?



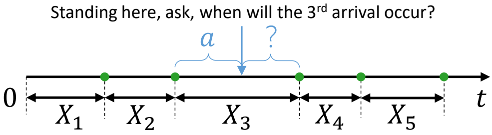
$$\mathbb{P}(X_3 - a > x | X_3 > a)$$

- **Question 1**: When will the next appear?



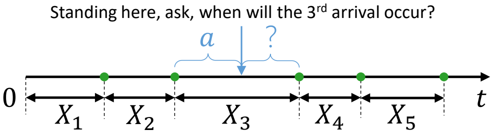
$$\mathbb{P}(X_3 - a > x | X_3 > a) = \frac{\mathbb{P}(X_3 - a > x, X_3 > a)}{\mathbb{P}(X_3 > a)}$$

- **Question 1**: When will the next appear?



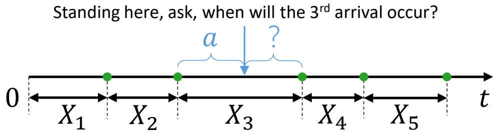
$$
\begin{aligned}
\mathbb{P}(X_3 - a > x | X_3 > a) &= \frac{\mathbb{P}(X_3 - a > x, X_3 > a)}{\mathbb{P}(X_3 > a)} \\
&= \frac{\mathbb{P}(X_3 > a + x, X_3 > a)}{\mathbb{P}(X_3 > a)}
\end{aligned}
$$

- **Question 1**: When will the next appear?



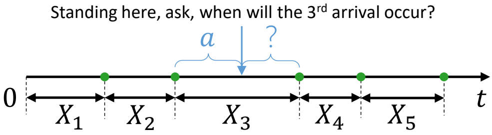
$$\mathbb{P}(X_3 - a > x | X_3 > a) = \frac{\mathbb{P}(X_3 - a > x, X_3 > a)}{\mathbb{P}(X_3 > a)}$$
$$= \frac{\mathbb{P}(X_3 > a + x, X_3 > a)}{\mathbb{P}(X_3 > a)}$$
$$= \frac{\mathbb{P}(X_3 > a + x)}{\mathbb{P}(X_3 > a)}$$

- **Question 1**: When will the next appear?



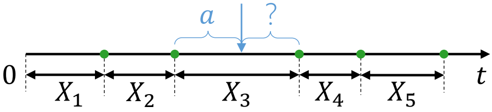
$$
\begin{aligned}
\mathbb{P}(X_3 - a > x | X_3 > a) &= \frac{\mathbb{P}(X_3 - a > x, X_3 > a)}{\mathbb{P}(X_3 > a)} \\
&= \frac{\mathbb{P}(X_3 > a + x, X_3 > a)}{\mathbb{P}(X_3 > a)} \\
&= \frac{\mathbb{P}(X_3 > a + x)}{\mathbb{P}(X_3 > a)} \\
&= \frac{e^{-\lambda(a+x)}}{e^{-\lambda a}} = e^{-\lambda x}.
\end{aligned}
$$

- **Question 1**: When will the next appear?



$$\begin{aligned}\mathbb{P}(X_3 - a > x | X_3 > a) &= \frac{\mathbb{P}(X_3 - a > x, X_3 > a)}{\mathbb{P}(X_3 > a)} \\ &= \frac{\mathbb{P}(X_3 > a + x, X_3 > a)}{\mathbb{P}(X_3 > a)} \\ &= \frac{\mathbb{P}(X_3 > a + x)}{\mathbb{P}(X_3 > a)} \\ &= \frac{e^{-\lambda(a+x)}}{e^{-\lambda a}} = e^{-\lambda x}. \quad \text{(Not related to } a\text{!)}\end{aligned}$$

- **Question 1**: When will the next appear?



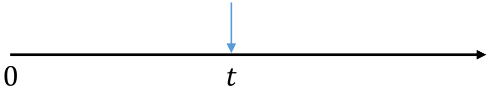
$$
\begin{aligned}
\mathbb{P}(X_3 - a > x | X_3 > a) &= \frac{\mathbb{P}(X_3 - a > x, X_3 > a)}{\mathbb{P}(X_3 > a)} \\
&= \frac{\mathbb{P}(X_3 > a + x, X_3 > a)}{\mathbb{P}(X_3 > a)} \\
&= \frac{\mathbb{P}(X_3 > a + x)}{\mathbb{P}(X_3 > a)} \\
&= \frac{e^{-\lambda(a+x)}}{e^{-\lambda a}} = e^{-\lambda x}. \quad \text{(Not related to } a\text{!)}
\end{aligned}
$$

- The Poisson process has no memory! (equivalent to the independent and stationary increments assumption)

- Let $S_n = X_1 + X_2 + \cdots + X_n$ be the arrival time of the nth arrival.
- **Question 2**: If I only know there are n arrivals up to time t, what can I say about the n arrival times $S_1, \ldots, S_n$?

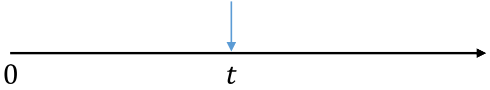
- Let $S_n = X_1 + X_2 + \cdots + X_n$ be the arrival time of the nth arrival.
- **Question 2**: If I only know there are n arrivals up to time t, what can I say about the n arrival times $S_1, \ldots, S_n$?
- A simplified case:

I'm only told that up to time t, one arrival has occurred.
What is the distribution of that arrival time S_1 $(= X_1)$?



- Let $S_n = X_1 + X_2 + \cdots + X_n$ be the arrival time of the nth arrival.
- **Question 2**: If I only know there are n arrivals up to time t, what can I say about the n arrival times $S_1, \ldots, S_n$?
- A simplified case:

I'm only told that up to time t, one arrival has occurred.
What is the distribution of that arrival time S_1 $(= X_1)$?



- Intuition:
  - Since Poisson process possesses independent and stationary increments, each interval of equal length in $[0, t]$ should have the same probability of containing the arrival.
  - Hence, the arrival time should be uniformly distributed on $[0, t]$.

Proof.

$$\mathbb{P}\{X_1 < s | N(t) = 1\}$$

Proof.

$$\mathbb{P}\{X_1 < s | N(t) = 1\} = \frac{\mathbb{P}\{X_1 < s, N(t) = 1\}}{\mathbb{P}\{N(t) = 1\}}$$

Proof.

$$\begin{aligned}\mathbb{P}\{X_1 < s | N(t) = 1\} &= \frac{\mathbb{P}\{X_1 < s, N(t) = 1\}}{\mathbb{P}\{N(t) = 1\}} \\ &= \frac{\mathbb{P}\{\text{1 arrival in } [0, s), \text{0 arrival in } [s, t)\}}{\mathbb{P}\{N(t) = 1\}}\end{aligned}$$

Proof.

$$\begin{aligned}\mathbb{P}\{X_1 < s | N(t) = 1\} &= \frac{\mathbb{P}\{X_1 < s, N(t) = 1\}}{\mathbb{P}\{N(t) = 1\}} \\ &= \frac{\mathbb{P}\{\text{1 arrival in } [0, s), \text{0 arrival in } [s, t)\}}{\mathbb{P}\{N(t) = 1\}} \\ &= \frac{\mathbb{P}\{\text{1 arrival in } [0, s)\}\,\mathbb{P}\{\text{0 arrival in } [s, t)\}}{\mathbb{P}\{N(t) = 1\}} \qquad \text{(independent)}\end{aligned}$$

Proof.

$$\begin{aligned}
\mathbb{P}\{X_1 < s | N(t) = 1\} &= \frac{\mathbb{P}\{X_1 < s, N(t) = 1\}}{\mathbb{P}\{N(t) = 1\}} \\
&= \frac{\mathbb{P}\{\text{1 arrival in } [0, s), \text{0 arrival in } [s, t)\}}{\mathbb{P}\{N(t) = 1\}} \\
&= \frac{\mathbb{P}\{\text{1 arrival in } [0, s)\}\,\mathbb{P}\{\text{0 arrival in } [s, t)\}}{\mathbb{P}\{N(t) = 1\}} \quad \text{(independent)} \\
&= \frac{\mathbb{P}\{N(s) = 1\}\,\mathbb{P}\{N(t - s) = 0\}}{\mathbb{P}\{N(t) = 1\}} \quad \text{(stationary)}
\end{aligned}$$

Proof.

$$\begin{aligned}
\mathbb{P}\{X_1 < s|N(t)=1\} &= \frac{\mathbb{P}\{X_1 < s, N(t)=1\}}{\mathbb{P}\{N(t)=1\}} \\
&= \frac{\mathbb{P}\{\text{1 arrival in } [0,s), \text{0 arrival in } [s,t)\}}{\mathbb{P}\{N(t)=1\}} \\
&= \frac{\mathbb{P}\{\text{1 arrival in } [0,s)\}\,\mathbb{P}\{\text{0 arrival in } [s,t)\}}{\mathbb{P}\{N(t)=1\}} \quad \text{(independent)} \\
&= \frac{\mathbb{P}\{N(s)=1\}\,\mathbb{P}\{N(t-s)=0\}}{\mathbb{P}\{N(t)=1\}} \quad \text{(stationary)} \\
&= \frac{e^{-\lambda s}\lambda s e^{-\lambda(t-s)}}{e^{-\lambda t}\lambda t}
\end{aligned}$$

上海交通大學 SHANGHAI JIAO TONG UNIVERSITY

Proof.

$$\begin{aligned}
\mathbb{P}\{X_1 < s | N(t) = 1\} &= \frac{\mathbb{P}\{X_1 < s, N(t) = 1\}}{\mathbb{P}\{N(t) = 1\}} \\
&= \frac{\mathbb{P}\{\text{1 arrival in } [0, s), \text{0 arrival in } [s, t)\}}{\mathbb{P}\{N(t) = 1\}} \\
&= \frac{\mathbb{P}\{\text{1 arrival in } [0, s)\}\, \mathbb{P}\{\text{0 arrival in } [s, t)\}}{\mathbb{P}\{N(t) = 1\}} \quad \text{(independent)} \\
&= \frac{\mathbb{P}\{N(s) = 1\}\, \mathbb{P}\{N(t - s) = 0\}}{\mathbb{P}\{N(t) = 1\}} \quad \text{(stationary)} \\
&= \frac{e^{-\lambda s} \lambda s e^{-\lambda(t-s)}}{e^{-\lambda t} \lambda t} \\
&= \frac{s}{t}.
\end{aligned}$$

■

Proof.

$$
\begin{aligned}
\mathbb{P}\{X_1 < s | N(t) = 1\} &= \frac{\mathbb{P}\{X_1 < s, N(t) = 1\}}{\mathbb{P}\{N(t) = 1\}} \\
&= \frac{\mathbb{P}\{\text{1 arrival in } [0, s), \text{0 arrival in } [s, t)\}}{\mathbb{P}\{N(t) = 1\}} \\
&= \frac{\mathbb{P}\{\text{1 arrival in } [0, s)\}\, \mathbb{P}\{\text{0 arrival in } [s, t)\}}{\mathbb{P}\{N(t) = 1\}} \quad \text{(independent)} \\
&= \frac{\mathbb{P}\{N(s) = 1\}\, \mathbb{P}\{N(t - s) = 0\}}{\mathbb{P}\{N(t) = 1\}} \quad \text{(stationary)} \\
&= \frac{e^{-\lambda s} \lambda s e^{-\lambda(t-s)}}{e^{-\lambda t} \lambda t} \\
&= \frac{s}{t}.
\end{aligned}
$$

■

- Remark: This result can be generalized to n arrivals.

Property (Conditional Distribution of Arrival Times)

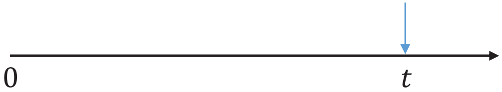
Given that $N(t) = n$, the n arrival times $S_1, \dots, S_n$ have the same distribution as the order statistics corresponding to n independent RVs uniformly distributed on the interval $(0, t)$.

Property (Conditional Distribution of Arrival Times)

Given that $N(t) = n$, the n arrival times $S_1, \ldots, S_n$ have the same distribution as the order statistics corresponding to n independent RVs uniformly distributed on the interval $(0, t)$.

- Illustration:

Given $N(t) = n$, how can I generate a sample of $\{S_1, S_2, \ldots, S_n\}$?



Property (Conditional Distribution of Arrival Times)

Given that $N(t) = n$, the n arrival times $S_1, \ldots, S_n$ have the same distribution as the order statistics corresponding to n independent RVs uniformly distributed on the interval $(0, t)$.

- Illustration:

Given $N(t) = n$, how can I generate a sample of $\{S_1, S_2, \ldots, S_n\}$?

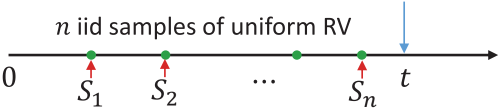
1. **Uniformly** and **independently** sample n points on $[0, t]$.

Property (Conditional Distribution of Arrival Times)

Given that $N(t) = n$, the n arrival times $S_1, \ldots, S_n$ have the same distribution as the order statistics corresponding to n independent RVs uniformly distributed on the interval $(0, t)$.

- Illustration:

Given $N(t) = n$, how can I generate a sample of $\{S_1, S_2, \ldots, S_n\}$?



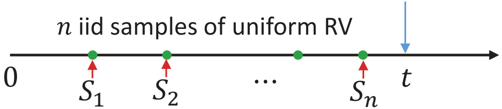
1. **Uniformly** and **independently** sample n points on $[0, t]$.
2. From small to large, call them $S_1, S_2, \ldots, S_n$.

Property (Conditional Distribution of Arrival Times)

Given that $N(t) = n$, the n arrival times $S_1, \ldots, S_n$ have the same distribution as the order statistics corresponding to n independent RVs uniformly distributed on the interval $(0, t)$.

- Illustration:

Given $N(t) = n$, how can I generate a sample of $\{S_1, S_2, \ldots, S_n\}$?



1. **Uniformly** and **independently** sample n points on $[0, t]$.
2. From small to large, call them $S_1, S_2, \ldots, S_n$.

- This is very nice for simulation!

- Let $L(t)$ denote the number of customers in the station at time t.

- Let $L(t)$ denote the number of customers in the station at time t.

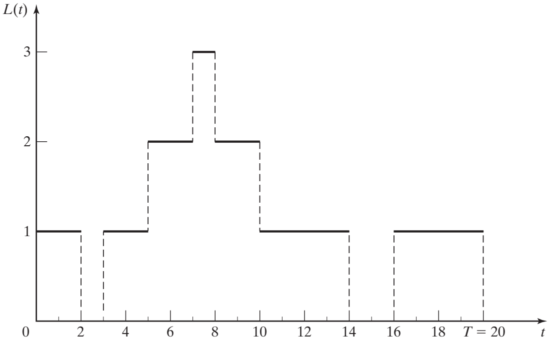


Figure: Illustration of $L(t)$ (*from Banks et al. (2010)*)

- Let $L(t)$ denote the number of customers in the station at time t.

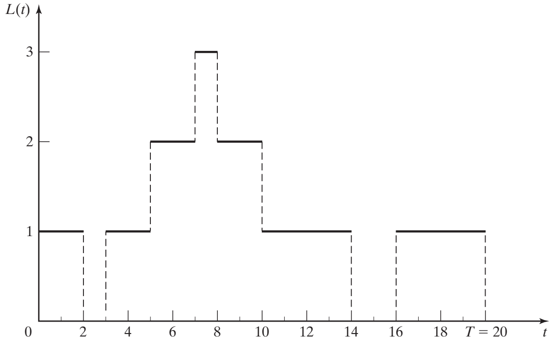


Figure: Illustration of $L(t)$ (*from Banks et al. (2010)*)

- Let $\widehat{L}(T)$ denote the (time-weighted) average number of customers in the station up to time T:

$$\widehat{L}(T) := \frac{1}{T}\int_0^T L(t)\mathrm{d}t.$$

- Another expression of $\widehat{L}(T)$: Let T_n denote the total time during $[0, T]$ in which the station contains exactly n customers.

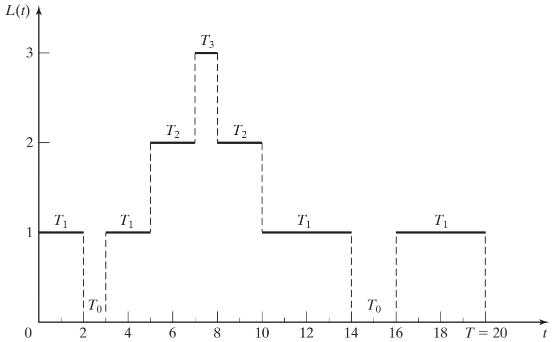


Figure: Illustration of $L(t)$ (*from Banks et al. (2010)*)

- $\widehat{L}(T) \coloneqq \frac{1}{T}\int_0^T L(t)\mathrm{d}t = \frac{1}{T}\sum_{n=0}^{\infty} nT_n = \sum_{n=0}^{\infty} n\left(\frac{T_n}{T}\right).$

- Suppose during time $[0, T]$, totally $N(T)$ customers have entered the station, and let $W_1, W_2, \ldots, W_{N(T)}$ denote the time each customer spends in the station up to time T.[†]

[†] The time includes both the waiting time in queue and the time in server. The part after T is not counted.

- Suppose during time $[0, T]$, totally $N(T)$ customers have entered the station, and let $W_1, W_2, \ldots, W_{N(T)}$ denote the time each customer spends in the station up to time T.[†]
- Let $\widehat{W}(T)$ denote the average sojourn time (逗留时间) in the station up to time T:

$$\widehat{W}(T) := \frac{1}{N(T)} \sum_{i=1}^{N(T)} W_i.$$

[†] The time includes both the waiting time in queue and the time in server. The part after T is not counted.

- Suppose during time $[0, T]$, totally $N(T)$ customers have entered the station, and let $W_1, W_2, \ldots, W_{N(T)}$ denote the time each customer spends in the station up to time T.†
- Let $\widehat{W}(T)$ denote the average sojourn time (逗留时间) in the station up to time T:

$$\widehat{W}(T) := \frac{1}{N(T)} \sum_{i=1}^{N(T)} W_i.$$

- In a similar way, we can also define
  - $\widehat{L}_Q(T)$ – The average number of customers in the *queue* up to time T.
  - $\widehat{W}_Q(T)$ – The average *waiting* time in the *queue* up to time T.

† The time includes both the waiting time in queue and the time in server. The part after T is not counted.

- Now we consider the long-run measures.
  - L – The long-run average number of customers in the station:
  $$L := \lim_{T\to\infty} \widehat{L}(T).$$
  - W – The long-run average sojourn time in the station:
  $$W := \lim_{T\to\infty} \widehat{W}(T).$$
  - L_Q – The long-run average number of customers in the queue:
  $$L_Q := \lim_{T\to\infty} \widehat{L}_Q(T).$$
  - W_Q – The long-run average waiting time in the queue:
  $$W_Q := \lim_{T\to\infty} \widehat{W}_Q(T).$$

- Now we consider the long-run measures.
  - L – The long-run average number of customers in the station:
  $$L := \lim_{T\to\infty} \widehat{L}(T).$$
  - W – The long-run average sojourn time in the station:
  $$W := \lim_{T\to\infty} \widehat{W}(T).$$
  - L_Q – The long-run average number of customers in the queue:
  $$L_Q := \lim_{T\to\infty} \widehat{L}_Q(T).$$
  - W_Q – The long-run average waiting time in the queue:
  $$W_Q := \lim_{T\to\infty} \widehat{W}_Q(T).$$
- Question: When will L, W, L_Q and W_Q exist (and $< \infty$)?

- We also define the *limiting probability* that there will be exactly n customers in the station as time goes to infinity:

$$P_n := \lim_{t\to\infty} \mathbb{P}\{L(t) = n\}, \quad n = 0, 1, 2, \ldots .$$

- We also define the *limiting probability* that there will be exactly n customers in the station as time goes to infinity:

$$P_n := \lim_{t\to\infty} \mathbb{P}\{L(t) = n\}, \quad n = 0, 1, 2, \ldots.$$

- Question: When will P_n exist?

- We also define the *limiting probability* that there will be exactly n customers in the station as time goes to infinity:

$$P_n := \lim_{t\to\infty} \mathbb{P}\{L(t) = n\}, \quad n = 0, 1, 2, \ldots.$$

- Question: When will P_n exist?

- Moreover, for an arbitrary $X/Y/s/K$ queue
  - Let λ denote the arrival rate, i.e.,

$$\mathbb{E}[\text{interarrival time}] = \frac{1}{\lambda}.$$

  - Let μ denote the service rate in one server, i.e.,

$$\mathbb{E}[\text{service time}] = \frac{1}{\mu}.$$

- Question: When will L, W, L_Q, W_Q and P_n exist?

- Question: When will L, W, L_Q, W_Q and P_n exist?
- Answer: When the queue is **stable**.†

† That is to say, the underlying Markov chain is positive recurrent.

- Question: When will L, W, L_Q, W_Q and P_n exist?
- Answer: When the queue is **stable**.[†]
- Question: When will the queue be stable?!

[†] That is to say, the underlying Markov chain is positive recurrent.

- Question: When will L, W, L_Q, W_Q and P_n exist?
- Answer: When the queue is **stable**.†
- Question: When will the queue be stable?!

Theorem 1 (Condition of Stability)

For an $X/Y/s/\infty$ queue (i.e., infinite capacity) with arrival rate λ and service rate μ, it is stable if

$$\lambda < s\mu.$$

And, an $X/Y/s/K$ queue (i.e., finite capacity) will always be stable.

† That is to say, the underlying Markov chain is positive recurrent.

- Recall that $P_n := \lim_{t\to\infty} \mathbb{P}\{L(t) = n\}, \ n = 0, 1, 2, \ldots$.

- Recall that $P_n := \lim_{t\to\infty} \mathbb{P}\{L(t) = n\},\ n = 0, 1, 2, \ldots.$
- P_n is also called the probability that there are exactly n customers in the station when it is in the *steady state*.
  - Since the system is stable and run for infinitely long time, it should enters some steady state (i.e., has nothing to do with the initial state).

上海交通大學 SHANGHAI JIAO TONG UNIVERSITY

- Recall that $P_n := \lim_{t\to\infty} \mathbb{P}\{L(t) = n\},\ n = 0, 1, 2, \ldots$.
- P_n is also called the probability that there are exactly n customers in the station when it is in the *steady state*.
  - Since the system is stable and run for infinitely long time, it should enters some steady state (i.e., has nothing to do with the initial state).
- L can also be written as $L := \sum_{n=0}^{\infty} nP_n$ (see next slide).
  - L is also called the expected number of customers in the station in steady state;
  - W is also called the expected sojourn time in the station in steady state;
  - L_Q is also called the expected number of customers in the queue in steady state;
  - W_Q is also called the expected waiting time in the queue in steady state.

- Recall that $P_n := \lim_{t\to\infty} \mathbb{P}\{L(t) = n\}, \ n = 0, 1, 2, \ldots$.

- Recall that $P_n := \lim_{t\to\infty} \mathbb{P}\{L(t) = n\}, \ n = 0, 1, 2, \ldots$.
- It turns out that, when the queue is *stable*, P_n also equals the *long-run proportion of time that the station contains exactly* n *customers*,[†] i.e., with probability 1, for all n,

$$P_n = \lim_{T\to\infty} \frac{\text{amount of time during } [0,T] \text{ that station contains } n \text{ customers}}{T}.$$

[†] A sufficient condition is that the queueing process is regenerative, which is satisfied in our discussion.

- Recall that $P_n := \lim_{t\to\infty} \mathbb{P}\{L(t) = n\},\ n = 0, 1, 2, \ldots.$
- It turns out that, when the queue is *stable*, P_n also equals the *long-run proportion of time that the station contains exactly n customers*,† i.e., with probability 1, for all n,

$$P_n = \lim_{T\to\infty} \frac{\text{amount of time during } [0, T] \text{ that station contains } n \text{ customers}}{T}.$$

- Recall $\widehat{L}(T) := \frac{1}{T}\int_0^T L(t)\mathrm{d}t = \sum_{n=0}^{\infty} n\left(\frac{T_n}{T}\right)$, then

† A sufficient condition is that the queueing process is regenerative, which is satisfied in our discussion.

- Recall that $P_n := \lim_{t\to\infty} \mathbb{P}\{L(t) = n\},\ n = 0, 1, 2, \ldots$.
- It turns out that, when the queue is *stable*, P_n also equals the *long-run proportion of time that the station contains exactly* n *customers*,† i.e., with probability 1, for all n,

$$P_n = \lim_{T\to\infty} \frac{\text{amount of time during } [0,T] \text{ that station contains } n \text{ customers}}{T}.$$

- Recall $\widehat{L}(T) := \frac{1}{T}\int_0^T L(t)\mathrm{d}t = \sum_{n=0}^{\infty} n\left(\frac{T_n}{T}\right)$, then

$$L := \lim_{T\to\infty} \widehat{L}(T) = \lim_{T\to\infty} \sum_{n=0}^{\infty} n\left(\frac{T_n}{T}\right)$$

† A sufficient condition is that the queueing process is regenerative, which is satisfied in our discussion.

- Recall that $P_n := \lim_{t\to\infty} \mathbb{P}\{L(t) = n\}$, $n = 0, 1, 2, \ldots$.
- It turns out that, when the queue is *stable*, P_n also equals the *long-run proportion of time that the station contains exactly* n *customers*,† i.e., with probability 1, for all n,

$$P_n = \lim_{T\to\infty} \frac{\text{amount of time during } [0,T] \text{ that station contains } n \text{ customers}}{T}.$$

- Recall $\widehat{L}(T) := \frac{1}{T}\int_0^T L(t)\mathrm{d}t = \sum_{n=0}^{\infty} n\left(\frac{T_n}{T}\right)$, then

$$L := \lim_{T\to\infty} \widehat{L}(T) = \lim_{T\to\infty} \sum_{n=0}^{\infty} n\left(\frac{T_n}{T}\right)$$

$$= \sum_{n=0}^{\infty} \lim_{T\to\infty} n\left(\frac{T_n}{T}\right) \quad \text{(by DCT)}$$

† A sufficient condition is that the queueing process is regenerative, which is satisfied in our discussion.

- Recall that $P_n := \lim_{t\to\infty} \mathbb{P}\{L(t) = n\},\ n = 0, 1, 2, \ldots$.
- It turns out that, when the queue is *stable*, P_n also equals the *long-run proportion of time that the station contains exactly* n *customers*,† i.e., with probability 1, for all n,

$$P_n = \lim_{T\to\infty} \frac{\text{amount of time during } [0,T] \text{ that station contains } n \text{ customers}}{T}.$$

- Recall $\widehat{L}(T) := \frac{1}{T}\int_0^T L(t)\mathrm{d}t = \sum_{n=0}^{\infty} n\left(\frac{T_n}{T}\right)$, then

$$\begin{aligned} L := \lim_{T\to\infty} \widehat{L}(T) &= \lim_{T\to\infty} \sum_{n=0}^{\infty} n\left(\frac{T_n}{T}\right) \\ &= \sum_{n=0}^{\infty} \lim_{T\to\infty} n\left(\frac{T_n}{T}\right) \quad \text{(by DCT)} \\ &= \sum_{n=0}^{\infty} nP_n. \end{aligned}$$

† A sufficient condition is that the queueing process is regenerative, which is satisfied in our discussion.

- Little's Law (守恒方程) is one of the most general and versatile laws in queueing theory.
  - It is named after John D.C. Little, who was the first to prove a version of it, in 1961.
  - When used in clever ways, Little's Law can lead to remarkably simple derivations.

- Little's Law (守恒方程) is one of the most general and versatile laws in queueing theory.
  - It is named after John D.C. Little, who was the first to prove a version of it, in 1961.
  - When used in clever ways, Little's Law can lead to remarkably simple derivations.

Theorem 2 (Little's Law – Empirical Version)

Define the observed entering rate $\widehat{\lambda} := N(T)/T$, then

$$\widehat{L}(T) = \widehat{\lambda}\widehat{W}(T), \quad \widehat{L}_Q(T) = \widehat{\lambda}\widehat{W}_Q(T).$$

- Verify Little's Law.

- Verify Little's Law.

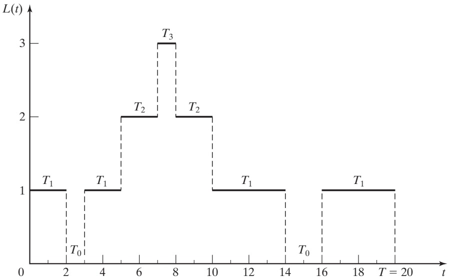


Figure: Illustration of $L(t)$ and W_i (*from* *Banks et al. (2010)*)

- Verify Little's Law.

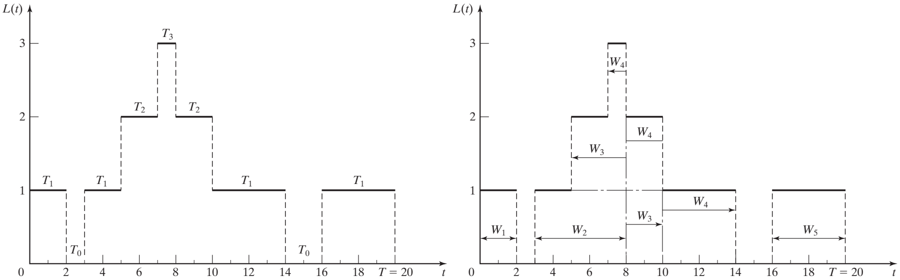


Figure: Illustration of $L(t)$ and W_i (*from* *Banks et al. (2010)*)

$$\widehat{\lambda} = N(T)/T = 5/20 = 0.25.$$

- Verify Little's Law.

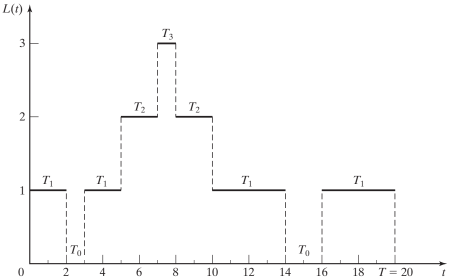


Figure: Illustration of $L(t)$ and W_i (*from Banks et al. (2010)*)

$$\widehat{\lambda} = N(T)/T = 5/20 = 0.25.$$

$$\widehat{W}(T) = \frac{1}{N(T)} \sum_{i=1}^{N(T)} W_i = \frac{1}{5}(2+5+5+7+4) = \frac{23}{5} = 4.6.$$

- Verify Little's Law.

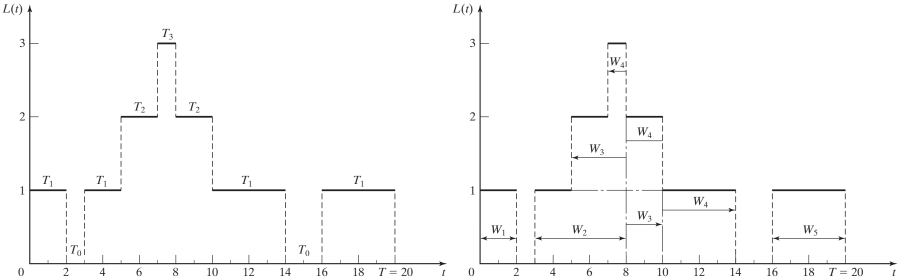


Figure: Illustration of $L(t)$ and W_i (*from Banks et al. (2010)*)

$$\widehat{\lambda} = N(T)/T = 5/20 = 0.25.$$

$$\widehat{W}(T) = \frac{1}{N(T)} \sum_{i=1}^{N(T)} W_i = \frac{1}{5}(2+5+5+7+4) = \frac{23}{5} = 4.6.$$

$$\widehat{L}(T) = \frac{1}{T} \sum_{n=0}^{\infty} nT_n = \frac{1}{20}(0 \times 3 + 1 \times 12 + 2 \times 4 + 3 \times 1) = \frac{23}{20} = 1.15.$$

- Verify Little's Law.

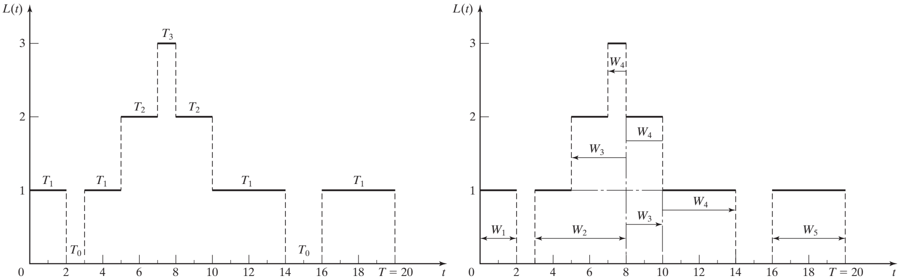


Figure: Illustration of $L(t)$ and W_i (*from Banks et al. (2010)*)

$$\widehat{\lambda} = N(T)/T = 5/20 = 0.25.$$

$$\widehat{W}(T) = \frac{1}{N(T)} \sum_{i=1}^{N(T)} W_i = \frac{1}{5}(2+5+5+7+4) = \frac{23}{5} = 4.6.$$

$$\widehat{L}(T) = \frac{1}{T} \sum_{n=0}^{\infty} nT_n = \frac{1}{20}(0 \times 3 + 1 \times 12 + 2 \times 4 + 3 \times 1) = \frac{23}{20} = 1.15.$$

So, $\widehat{\lambda}\widehat{W}(T) = 0.25 \times 4.6 = 1.15 = \widehat{L}(T)$.

- Verify Little's Law.

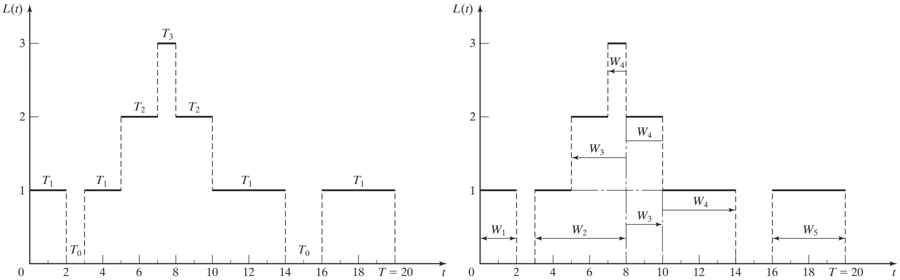


Figure: Illustration of $L(t)$ and W_i (*from* *Banks et al. (2010)*)

$\widehat{\lambda} = N(T)/T = 5/20 = 0.25.$

$\widehat{W}(T) = \frac{1}{N(T)} \sum_{i=1}^{N(T)} W_i = \frac{1}{5}(2+5+5+7+4) = \frac{23}{5} = 4.6.$

$\widehat{L}(T) = \frac{1}{T} \sum_{n=0}^{\infty} nT_n = \frac{1}{20}(0\times 3 + 1\times 12 + 2\times 4 + 3\times 1) = \frac{23}{20} = 1.15.$

So, $\widehat{\lambda}\widehat{W}(T) = 0.25 \times 4.6 = 1.15 = \widehat{L}(T).$ **(Why it always holds?)**

- Verify Little's Law.

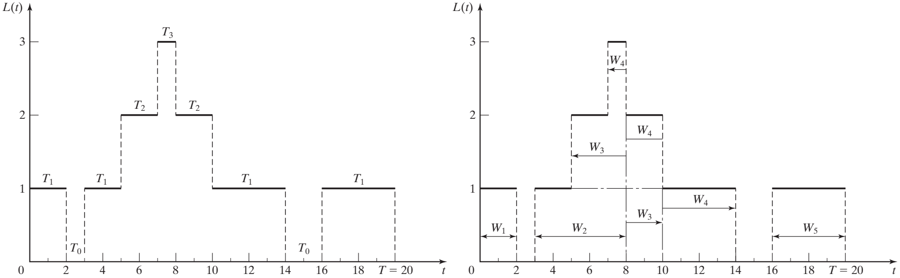


Figure: Illustration of $L(t)$ and W_i (*from Banks et al. (2010)*)

- **Why it always holds?**

- Verify Little's Law.

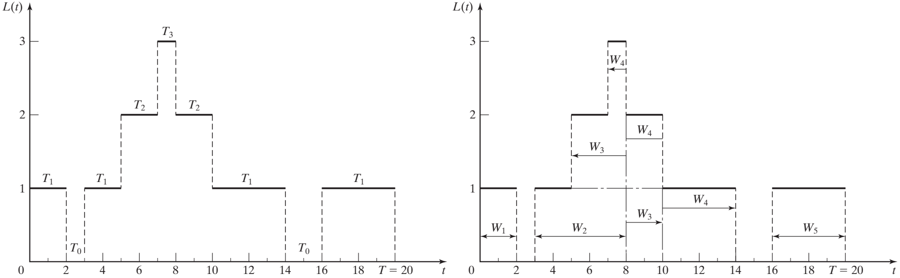


Figure: Illustration of $L(t)$ and W_i (*from Banks et al. (2010)*)

- **Why it always holds?**

$\widehat{L}(T) = \frac{1}{T}\sum_{n=0}^{\infty} nT_n = \frac{1}{T} \times$ area.

- Verify Little's Law.

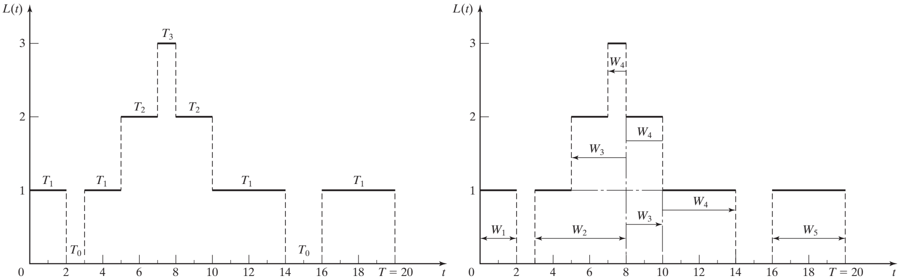


Figure: Illustration of $L(t)$ and W_i (*from* *Banks et al. (2010)*)

- **Why it always holds?**

$\widehat{L}(T) = \frac{1}{T}\sum_{n=0}^{\infty} nT_n = \frac{1}{T} \times$ area.

$\widehat{\lambda}\widehat{W}(T) = \frac{N(T)}{T}\frac{1}{N(T)}\sum_{i=1}^{N(T)} W_i$

- Verify Little's Law.

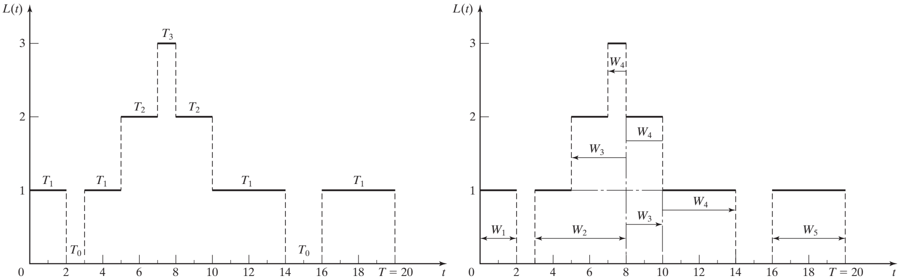


Figure: Illustration of $L(t)$ and W_i (*from* *Banks et al. (2010)*)

- **Why it always holds?**

$\widehat{L}(T) = \frac{1}{T}\sum_{n=0}^{\infty} nT_n = \frac{1}{T} \times$ area.

$\widehat{\lambda}\widehat{W}(T) = \frac{N(T)}{T}\frac{1}{N(T)}\sum_{i=1}^{N(T)} W_i = \frac{1}{T}\sum_{i=1}^{N(T)} W_i$

- Verify Little's Law.

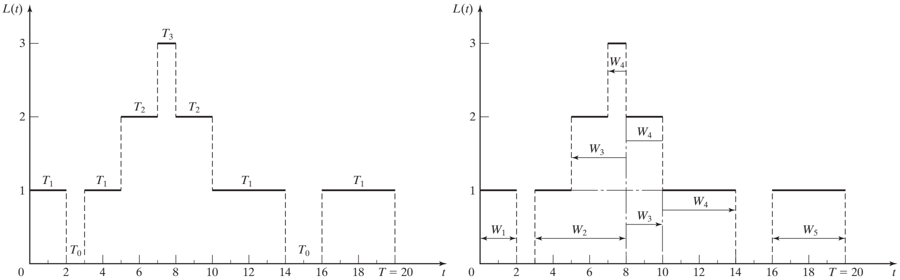


Figure: Illustration of $L(t)$ and W_i (*from* *Banks et al. (2010)*)

- **Why it always holds?**

$\widehat{L}(T) = \frac{1}{T}\sum_{n=0}^{\infty} nT_n = \frac{1}{T} \times$ area.

$\widehat{\lambda}\widehat{W}(T) = \frac{N(T)}{T}\frac{1}{N(T)}\sum_{i=1}^{N(T)} W_i = \frac{1}{T}\sum_{i=1}^{N(T)} W_i = \frac{1}{T} \times$ area.

- Verify Little's Law.

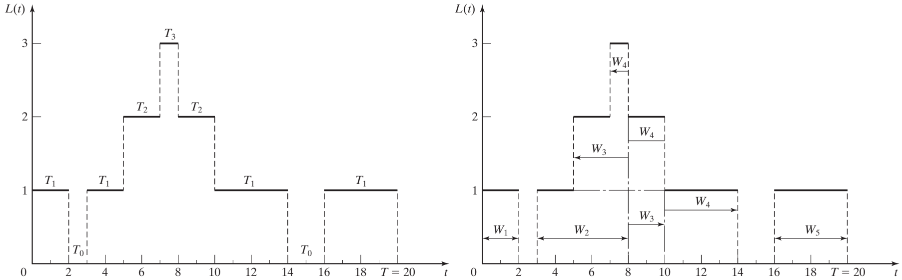


Figure: Illustration of $L(t)$ and W_i (*from* *Banks et al. (2010)*)

- **Why it always holds?**

$\widehat{L}(T) = \frac{1}{T}\sum_{n=0}^{\infty} nT_n = \frac{1}{T} \times$ area.

$\widehat{\lambda}\widehat{W}(T) = \frac{N(T)}{T}\frac{1}{N(T)}\sum_{i=1}^{N(T)} W_i = \frac{1}{T}\sum_{i=1}^{N(T)} W_i = \frac{1}{T} \times$ area.

So, $\widehat{L}(T) = \widehat{\lambda}\widehat{W}(T)$ always holds.

- Verify Little's Law.

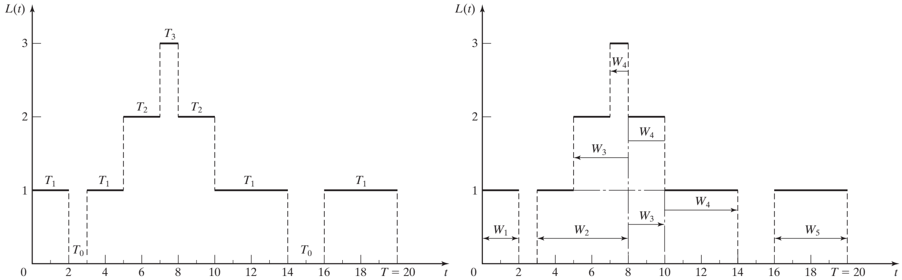


Figure: Illustration of $L(t)$ and W_i (*from* Banks et al. (2010))

- **Why it always holds?**

  $\widehat{L}(T) = \frac{1}{T}\sum_{n=0}^{\infty} nT_n = \frac{1}{T} \times$ area.

  $\widehat{\lambda}\widehat{W}(T) = \frac{N(T)}{T}\frac{1}{N(T)}\sum_{i=1}^{N(T)} W_i = \frac{1}{T}\sum_{i=1}^{N(T)} W_i = \frac{1}{T} \times$ area.

  So, $\widehat{L}(T) = \widehat{\lambda}\widehat{W}(T)$ always holds.

- **The same argument for $\widehat{L}_Q(T) = \widehat{\lambda}\widehat{W}_Q(T)$.**

Theorem 3 (Little's Law – Limit/Expectation Version)

For a stable queue, let λ^* denote the arrival rate or entering rate, then

$$L = \lambda^* W, \quad L_Q = \lambda^* W_Q.$$

Caution: When λ^* is the arrival rate, the time average (W, W_Q) is based on all customers (who enters the station and who are lost); When λ^* is the entering rate, the time average is only based on the customers who enters the station.

Theorem 3 (Little's Law – Limit/Expectation Version)

For a stable queue, let λ^* denote the arrival rate or entering rate, then

$$L = \lambda^* W, \quad L_Q = \lambda^* W_Q.$$

Caution: When λ^* is the arrival rate, the time average (W, W_Q) is based on all customers (who enters the station and who are lost); When λ^* is the entering rate, the time average is only based on the customers who enters the station.

- Some Remarks:
  - For a customer who is lost (due to the finite capacity), he spends 0 amount of time in the station (or queue).
  - Once we know anyone of L, W, L_Q and W_Q, we can compute the rest using Little's Law.

- $M/M/1$ Queue[†]
  - The interarrival times are iid random variables with $\mathrm{Exp}(\lambda)$ distribution, that is to say, *customers arrive according to a Poisson process with rate* λ.
  - The service times are iid random variables with $\mathrm{Exp}(\mu)$ distribution.
  - The customers are served in an FCFS fashion by a *single* server.
  - The capacity is unlimited, i.e., waiting space is unlimited.
  - $M/M/1$ queue is stable **if and only if** $\lambda < \mu$.
  - Due to unlimited capacity, arrival rate $=$ entering rate.

[†] $M/M/1$ Queue $\subset$ *Birth and Death Process with Infinite Capacity* $\subset$ Continuous-Time Markov Chain.

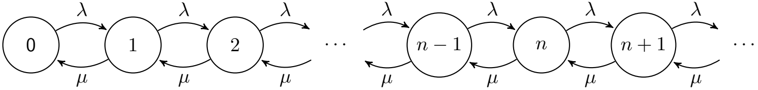
- $M/M/1$ Queue†
  - The interarrival times are iid random variables with $\mathrm{Exp}(\lambda)$ distribution, that is to say, *customers arrive according to a Poisson process with rate* λ.
  - The service times are iid random variables with $\mathrm{Exp}(\mu)$ distribution.
  - The customers are served in an FCFS fashion by a *single* server.
  - The capacity is unlimited, i.e., waiting space is unlimited.
  - $M/M/1$ queue is stable **if and only if** $\lambda < \mu$.
  - Due to unlimited capacity, arrival rate $=$ entering rate.
- We now want to compute all the measures P_n, L, W, L_Q and W_Q.

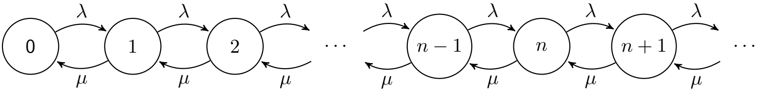
† $M/M/1$ Queue $\subset$ *Birth and Death Process with Infinite Capacity* $\subset$ Continuous-Time Markov Chain.

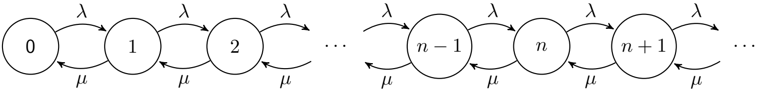
- Recall that L can be computed via $L = \sum_{n=0}^{\infty} nP_n$, where P_n has two interpretations:
  - Long-run proportion of time that the station contains exactly n customers;
  - Probability that there are exactly n customers in the station as time goes to infinity (or equivalently, in the steady state).

- Recall that L can be computed via $L = \sum_{n=0}^{\infty} nP_n$, where P_n has two interpretations:
  - Long-run proportion of time that the station contains exactly n customers;
  - Probability that there are exactly n customers in the station as time goes to infinity (or equivalently, in the steady state).
- Define the **state** as the the number of customers in the system.

- Recall that L can be computed via $L = \sum_{n=0}^{\infty} nP_n$, where P_n has two interpretations:
  - Long-run proportion of time that the station contains exactly n customers;
  - Probability that there are exactly n customers in the station as time goes to infinity (or equivalently, in the steady state).
- Define the **state** as the the number of customers in the system.
- The state space diagram is as follows:

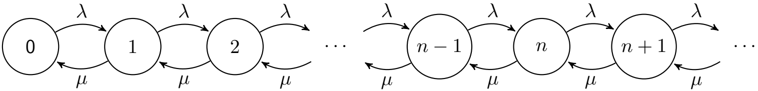



0
1
2
$\cdots$
$n-1$
n
$n+1$
$\cdots$
λ
μ



Key Observation 1

Rate at which the process leaves state n
= Rate at which the process enters state n.

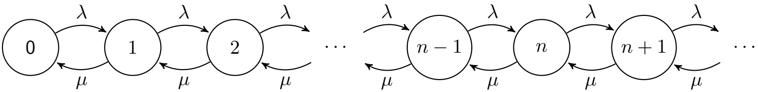


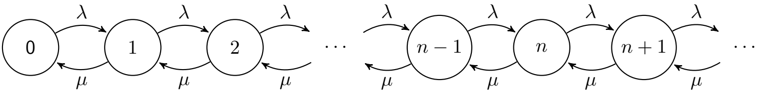
Key Observation 1

Rate at which the process leaves state n
$=$ Rate at which the process enters state n.

Heuristic Proof.

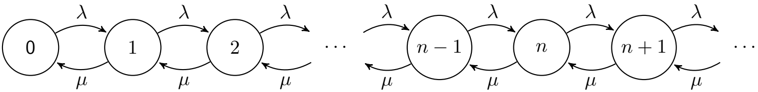
- In any time interval, the number of transitions into state n must equal to within 1 the number of transitions out of state n. (Why?)
- Hence, in the long run, the rate into state n must equal the rate out of state n.


0
1
2
$\cdots$
$n-1$
n
$n+1$
$\cdots$
λ
μ



Key Observation 2

Rate at which the process leaves state $0 = P_0\lambda$;
Rate at which the process leaves state $n = P_n(\mu+\lambda)$, $n \geq 1$;
Rate at which the process enters state $0 = P_1\mu$;
Rate at which the process enters state $n = P_{n-1}\lambda + P_{n+1}\mu$, $n \geq 1$.



Key Observation 2

Rate at which the process leaves state $0 = P_0\lambda$;
Rate at which the process leaves state $n = P_n(\mu+\lambda)$, $n \geq 1$;
Rate at which the process enters state $0 = P_1\mu$;
Rate at which the process enters state $n = P_{n-1}\lambda + P_{n+1}\mu$, $n \geq 1$.

Fact

If $X_1, \ldots, X_n$ are independent random variables, and $X_i \sim \mathrm{Exp}(\lambda_i)$, $i = 1, \ldots, n$, then

$$\min\{X_1, \ldots, X_n\} \sim \mathrm{Exp}(\lambda_1 + \cdots + \lambda_n).$$

Theorem 4 (Limiting Distribution of $M/M/1$ Queue)

For an $M/M/1$ queue, when it is stable ($\lambda < \mu$), its limiting (steady-state) distribution is given by

$$P_n = (1-\rho)\rho^n, \quad n \geq 0,$$

where $\rho := \lambda/\mu < 1$. (ρ is called the *server utilization*.)

Theorem 4 (Limiting Distribution of $M/M/1$ Queue)

For an $M/M/1$ queue, when it is stable ($\lambda < \mu$), its limiting (steady-state) distribution is given by

$$P_n = (1-\rho)\rho^n, \quad n \geq 0,$$

where $\rho := \lambda/\mu < 1$. (ρ is called the *server utilization*.)

Proof.

Theorem 4 (Limiting Distribution of $M/M/1$ Queue)

For an $M/M/1$ queue, when it is stable ($\lambda < \mu$), its limiting (steady-state) distribution is given by

$$P_n = (1-\rho)\rho^n, \quad n \geq 0,$$

where $\rho := \lambda/\mu < 1$. (ρ is called the *server utilization*.)

Proof. Due to Observations 1 & 2,

State	Rate Process Leaves		Rate Process Enters
0	$P_0\lambda$	$=$	$P_1\mu$
n, $n \geq 1$	$P_n(\mu+\lambda)$	$=$	$P_{n-1}\lambda + P_{n+1}\mu$

Theorem 4 (Limiting Distribution of $M/M/1$ Queue)

For an $M/M/1$ queue, when it is stable ($\lambda < \mu$), its limiting (steady-state) distribution is given by

$$P_n = (1-\rho)\rho^n, \quad n \geq 0,$$

where $\rho := \lambda/\mu < 1$. (ρ is called the *server utilization*.)

Proof. Due to Observations 1 & 2,

State	Rate Process Leaves		Rate Process Enters
0	$P_0\lambda$	$=$	$P_1\mu$
n, $n \geq 1$	$P_n(\mu+\lambda)$	$=$	$P_{n-1}\lambda + P_{n+1}\mu$

Rewriting these equations gives

$$P_0\lambda = P_1\mu,$$
$$P_n\lambda = P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu), \quad n \geq 1.$$

Recall that

$$
\begin{aligned}
P_0\lambda &= P_1\mu, \\
P_n\lambda &= P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu), \quad n \geq 1.
\end{aligned}
$$

Recall that

$$P_0\lambda = P_1\mu,$$
$$P_n\lambda = P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu), \quad n \geq 1.$$

Or, equivalently,

$$P_0\lambda = P_1\mu,$$
$$P_1\lambda = P_2\mu + (P_0\lambda - P_1\mu) = P_2\mu,$$

Recall that

$$P_0\lambda = P_1\mu,$$
$$P_n\lambda = P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu), \quad n \geq 1.$$

Or, equivalently,

$$P_0\lambda = P_1\mu,$$
$$P_1\lambda = P_2\mu + (P_0\lambda - P_1\mu) = P_2\mu,$$
$$P_2\lambda = P_3\mu + (P_1\lambda - P_2\mu) = P_3\mu,$$

Recall that

$$\begin{aligned} P_0\lambda &= P_1\mu, \\ P_n\lambda &= P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu), \quad n \geq 1. \end{aligned}$$

Or, equivalently,

$$\begin{aligned} P_0\lambda &= P_1\mu, \\ P_1\lambda &= P_2\mu + (P_0\lambda - P_1\mu) = P_2\mu, \\ P_2\lambda &= P_3\mu + (P_1\lambda - P_2\mu) = P_3\mu, \\ P_n\lambda &= P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu) = P_{n+1}\mu, \quad n \geq 1. \end{aligned}$$

Recall that

$$P_0\lambda = P_1\mu,$$
$$P_n\lambda = P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu), \quad n \geq 1.$$

Or, equivalently,

$$P_0\lambda = P_1\mu,$$
$$P_1\lambda = P_2\mu + (P_0\lambda - P_1\mu) = P_2\mu,$$
$$P_2\lambda = P_3\mu + (P_1\lambda - P_2\mu) = P_3\mu,$$
$$P_n\lambda = P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu) = P_{n+1}\mu, \quad n \geq 1.$$

Let $\rho := \lambda/\mu$ (< 1), solving in terms of P_0 yields

$$P_1 = P_0\rho,$$

Recall that

$$P_0\lambda = P_1\mu,$$
$$P_n\lambda = P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu), \quad n \geq 1.$$

Or, equivalently,

$$P_0\lambda = P_1\mu,$$
$$P_1\lambda = P_2\mu + (P_0\lambda - P_1\mu) = P_2\mu,$$
$$P_2\lambda = P_3\mu + (P_1\lambda - P_2\mu) = P_3\mu,$$
$$P_n\lambda = P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu) = P_{n+1}\mu, \quad n \geq 1.$$

Let $\rho := \lambda/\mu$ (< 1), solving in terms of P_0 yields

$$P_1 = P_0\rho,$$
$$P_2 = P_1\rho = P_0\rho^2,$$

Recall that

$$P_0\lambda = P_1\mu,$$
$$P_n\lambda = P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu), \quad n \geq 1.$$

Or, equivalently,

$$P_0\lambda = P_1\mu,$$
$$P_1\lambda = P_2\mu + (P_0\lambda - P_1\mu) = P_2\mu,$$
$$P_2\lambda = P_3\mu + (P_1\lambda - P_2\mu) = P_3\mu,$$
$$P_n\lambda = P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu) = P_{n+1}\mu, \quad n \geq 1.$$

Let $\rho := \lambda/\mu$ (< 1), solving in terms of P_0 yields

$$P_1 = P_0\rho,$$
$$P_2 = P_1\rho = P_0\rho^2,$$
$$P_n = P_{n-1}\rho = P_0\rho^n, \quad n \geq 1.$$

Recall that

$$P_0\lambda = P_1\mu,$$
$$P_n\lambda = P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu), \quad n \geq 1.$$

Or, equivalently,

$$P_0\lambda = P_1\mu,$$
$$P_1\lambda = P_2\mu + (P_0\lambda - P_1\mu) = P_2\mu,$$
$$P_2\lambda = P_3\mu + (P_1\lambda - P_2\mu) = P_3\mu,$$
$$P_n\lambda = P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu) = P_{n+1}\mu, \quad n \geq 1.$$

Let $\rho := \lambda/\mu$ (< 1), solving in terms of P_0 yields

$$P_1 = P_0\rho,$$
$$P_2 = P_1\rho = P_0\rho^2,$$
$$P_n = P_{n-1}\rho = P_0\rho^n, \quad n \geq 1.$$

Since $1 = \Sigma_{n=0}^{\infty} P_n = P_0 \Sigma_{n=0}^{\infty} \rho^n = P_0/(1-\rho)$, we have

Recall that

$$P_0\lambda = P_1\mu,$$
$$P_n\lambda = P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu), \quad n \geq 1.$$

Or, equivalently,

$$P_0\lambda = P_1\mu,$$
$$P_1\lambda = P_2\mu + (P_0\lambda - P_1\mu) = P_2\mu,$$
$$P_2\lambda = P_3\mu + (P_1\lambda - P_2\mu) = P_3\mu,$$
$$P_n\lambda = P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu) = P_{n+1}\mu, \quad n \geq 1.$$

Let $\rho \coloneqq \lambda/\mu$ (< 1), solving in terms of P_0 yields

$$P_1 = P_0\rho,$$
$$P_2 = P_1\rho = P_0\rho^2,$$
$$P_n = P_{n-1}\rho = P_0\rho^n, \quad n \geq 1.$$

Since $1 = \Sigma_{n=0}^{\infty} P_n = P_0 \Sigma_{n=0}^{\infty} \rho^n = P_0/(1-\rho)$, we have

$$P_0 = 1 - \rho, \quad \text{and} \quad P_n = (1-\rho)\rho^n, \quad n \geq 1. \quad \blacksquare$$

- $L = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = \frac{\rho}{1-\rho}$.

- $L = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = \frac{\rho}{1-\rho}$.
- Using Little's Law, $W = L/\lambda = \frac{1}{\lambda}\frac{\rho}{1-\rho} = \frac{1}{\mu-\lambda}$.

- $L = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = \frac{\rho}{1-\rho}$.
- Using Little's Law, $W = L/\lambda = \frac{1}{\lambda}\frac{\rho}{1-\rho} = \frac{1}{\mu-\lambda}$.
- $L_Q = \sum_{n=1}^{\infty}(n-1)P_n = \sum_{n=1}^{\infty}(n-1)(1-\rho)\rho^n = \frac{\rho^2}{1-\rho}$.

- $L = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = \frac{\rho}{1-\rho}$.
- Using Little's Law, $W = L/\lambda = \frac{1}{\lambda}\frac{\rho}{1-\rho} = \frac{1}{\mu-\lambda}$.
- $L_Q = \sum_{n=1}^{\infty}(n-1)P_n = \sum_{n=1}^{\infty}(n-1)(1-\rho)\rho^n = \frac{\rho^2}{1-\rho}$.
- Using Little's Law, $W_Q = L_Q/\lambda = \frac{1}{\lambda}\frac{\rho^2}{1-\rho} = \frac{1}{\mu}\frac{\rho}{1-\rho} = \frac{\rho}{\mu-\lambda}$.

- $L = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = \frac{\rho}{1-\rho}$.
- Using Little's Law, $W = L/\lambda = \frac{1}{\lambda}\frac{\rho}{1-\rho} = \frac{1}{\mu-\lambda}$.
- $L_Q = \sum_{n=1}^{\infty}(n-1)P_n = \sum_{n=1}^{\infty}(n-1)(1-\rho)\rho^n = \frac{\rho^2}{1-\rho}$.
- Using Little's Law, $W_Q = L_Q/\lambda = \frac{1}{\lambda}\frac{\rho^2}{1-\rho} = \frac{1}{\mu}\frac{\rho}{1-\rho} = \frac{\rho}{\mu-\lambda}$.
- Or, $W_Q = W - \mathbb{E}[\text{service time}] = \frac{1}{\mu-\lambda} - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{\mu-\lambda}$.

- $L = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = \frac{\rho}{1-\rho}$.
- Using Little's Law, $W = L/\lambda = \frac{1}{\lambda}\frac{\rho}{1-\rho} = \frac{1}{\mu-\lambda}$.
- $L_Q = \sum_{n=1}^{\infty}(n-1)P_n = \sum_{n=1}^{\infty}(n-1)(1-\rho)\rho^n = \frac{\rho^2}{1-\rho}$.
- Using Little's Law, $W_Q = L_Q/\lambda = \frac{1}{\lambda}\frac{\rho^2}{1-\rho} = \frac{1}{\mu}\frac{\rho}{1-\rho} = \frac{\rho}{\mu-\lambda}$.
- Or, $W_Q = W - \mathbb{E}[\text{service time}] = \frac{1}{\mu-\lambda} - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{\mu-\lambda}$.
- Using Little's Law, $L_Q = \lambda W_Q = \lambda\frac{\rho}{\mu-\lambda} = \frac{\rho^2}{1-\rho}$.

- $L = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = \frac{\rho}{1-\rho}$.
- Using Little's Law, $W = L/\lambda = \frac{1}{\lambda}\frac{\rho}{1-\rho} = \frac{1}{\mu-\lambda}$.
- $L_Q = \sum_{n=1}^{\infty}(n-1)P_n = \sum_{n=1}^{\infty}(n-1)(1-\rho)\rho^n = \frac{\rho^2}{1-\rho}$.
- Using Little's Law, $W_Q = L_Q/\lambda = \frac{1}{\lambda}\frac{\rho^2}{1-\rho} = \frac{1}{\mu}\frac{\rho}{1-\rho} = \frac{\rho}{\mu-\lambda}$.
- Or, $W_Q = W - \mathbb{E}[\text{service time}] = \frac{1}{\mu-\lambda} - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{\mu-\lambda}$.
- Using Little's Law, $L_Q = \lambda W_Q = \lambda\frac{\rho}{\mu-\lambda} = \frac{\rho^2}{1-\rho}$.
- Due to unlimited capacity, arrival rate = entering rate, so the time average (W, W_Q) is based on all customers.

- $L = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = \frac{\rho}{1-\rho}$.
- Using Little's Law, $W = L/\lambda = \frac{1}{\lambda}\frac{\rho}{1-\rho} = \frac{1}{\mu-\lambda}$.
- $L_Q = \sum_{n=1}^{\infty}(n-1)P_n = \sum_{n=1}^{\infty}(n-1)(1-\rho)\rho^n = \frac{\rho^2}{1-\rho}$.
- Using Little's Law, $W_Q = L_Q/\lambda = \frac{1}{\lambda}\frac{\rho^2}{1-\rho} = \frac{1}{\mu}\frac{\rho}{1-\rho} = \frac{\rho}{\mu-\lambda}$.
- Or, $W_Q = W - \mathbb{E}[\text{service time}] = \frac{1}{\mu-\lambda} - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{\mu-\lambda}$.
- Using Little's Law, $L_Q = \lambda W_Q = \lambda\frac{\rho}{\mu-\lambda} = \frac{\rho^2}{1-\rho}$.
- Due to unlimited capacity, arrival rate = entering rate, so the time average (W, W_Q) is based on all customers.
- $\mathbb{P}(\text{the server is idle}) = P_0 = 1-\rho$.

- $L = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = \frac{\rho}{1-\rho}$.
- Using Little's Law, $W = L/\lambda = \frac{1}{\lambda}\frac{\rho}{1-\rho} = \frac{1}{\mu-\lambda}$.
- $L_Q = \sum_{n=1}^{\infty}(n-1)P_n = \sum_{n=1}^{\infty}(n-1)(1-\rho)\rho^n = \frac{\rho^2}{1-\rho}$.
- Using Little's Law, $W_Q = L_Q/\lambda = \frac{1}{\lambda}\frac{\rho^2}{1-\rho} = \frac{1}{\mu}\frac{\rho}{1-\rho} = \frac{\rho}{\mu-\lambda}$.
- Or, $W_Q = W - \mathbb{E}[\text{service time}] = \frac{1}{\mu-\lambda} - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{\mu-\lambda}$.
- Using Little's Law, $L_Q = \lambda W_Q = \lambda\frac{\rho}{\mu-\lambda} = \frac{\rho^2}{1-\rho}$.
- Due to unlimited capacity, arrival rate = entering rate, so the time average (W, W_Q) is based on all customers.
- $\mathbb{P}(\text{the server is idle}) = P_0 = 1-\rho$.
- As $\rho \to 1$, all L, W, L_Q and W_Q tend to ∞.

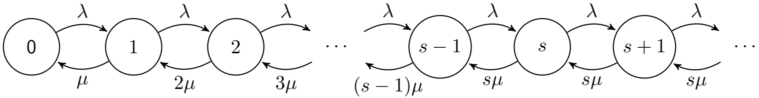
- $M/M/s$ Queue[†]
  - Customers arrive according to a Poisson process with rate λ.
  - The service times are iid random variables with $\mathrm{Exp}(\mu)$ distribution.
  - There are s parallel servers.
  - The customers form a single queue and get served by the next available server in an FCFS fashion.
  - The capacity is unlimited, i.e., waiting space is unlimited.
  - $M/M/s$ queue is stable **if and only if** $\lambda < s\mu$.
  - Due to unlimited capacity, arrival rate $=$ entering rate.

[†] $M/M/1$ Queue $\subset$ $M/M/s$ Queue $\subset$ *Birth and Death Process with Infinite Capacity* $\subset$ CTMC.

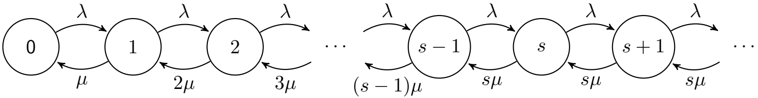
- $M/M/s$ Queue[†]
    - Customers arrive according to a Poisson process with rate λ.
    - The service times are iid random variables with $\text{Exp}(\mu)$ distribution.
    - There are s parallel servers.
    - The customers form a single queue and get served by the next available server in an FCFS fashion.
    - The capacity is unlimited, i.e., waiting space is unlimited.
    - $M/M/s$ queue is stable **if and only if** $\lambda < s\mu$.
    - Due to unlimited capacity, arrival rate $=$ entering rate.
- $M/M/s$ queue is a generalized version of $M/M/1$ queue. Let $s = 1$, all results should degenerate to those of $M/M/1$.

[†] $M/M/1$ Queue $\subset$ $M/M/s$ Queue $\subset$ *Birth and Death Process with Infinite Capacity* $\subset$ CTMC.

- The state space diagram is as follows:



- The state space diagram is as follows:



Theorem 5 (Limiting Distribution of $M/M/s$ Queue)

For an $M/M/s$ queue, when it is stable ($\lambda < s\mu$), its limiting (steady-state) distribution is given by

$$P_n = \left[\sum_{i=0}^{s} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i + \frac{s^s}{s!}\frac{\rho^{s+1}}{1-\rho}\right]^{-1} \rho_n\ , \quad n \geq 0,$$

where the *server utilization* $\rho := \lambda/(s\mu) < 1$, and

$$\rho_n := \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n, & \text{if } 0 \leq n \leq s, \\ \frac{s^s}{s!}\rho^n, & \text{if } n \geq s+1. \end{cases}$$

- $L_Q = \sum_{n=s}^{\infty}(n-s)P_n$

- $L_Q = \sum_{n=s}^{\infty}(n-s)P_n = \sum_{n=s}^{\infty}(n-s)P_0\rho_n$

- $L_Q = \sum_{n=s}^{\infty}(n-s)P_n = \sum_{n=s}^{\infty}(n-s)P_0\rho_n = \sum_{k=0}^{\infty} kP_0\rho_{s+k}$

- $L_Q = \sum_{n=s}^{\infty}(n-s)P_n = \sum_{n=s}^{\infty}(n-s)P_0\rho_n = \sum_{k=0}^{\infty} kP_0\rho_{s+k}$
  $= \sum_{k=1}^{\infty} kP_0\rho_s\rho^k$

- $L_Q = \sum_{n=s}^{\infty}(n-s)P_n = \sum_{n=s}^{\infty}(n-s)P_0\rho_n = \sum_{k=0}^{\infty} kP_0\rho_{s+k}$
  $= \sum_{k=1}^{\infty} kP_0\rho_s\rho^k = \sum_{k=1}^{\infty} kP_s\rho^k$

- $L_Q = \sum_{n=s}^{\infty}(n-s)P_n = \sum_{n=s}^{\infty}(n-s)P_0\rho_n = \sum_{k=0}^{\infty} kP_0\rho_{s+k}$
  $= \sum_{k=1}^{\infty} kP_0\rho_s\rho^k = \sum_{k=1}^{\infty} kP_s\rho^k = \frac{P_s\rho}{(1-\rho)^2}$.

- $L_Q = \sum_{n=s}^{\infty}(n-s)P_n = \sum_{n=s}^{\infty}(n-s)P_0\rho_n = \sum_{k=0}^{\infty} kP_0\rho_{s+k}$
  $= \sum_{k=1}^{\infty} kP_0\rho_s\rho^k = \sum_{k=1}^{\infty} kP_s\rho^k = \frac{P_s\rho}{(1-\rho)^2}$.
- Using Little's Law, $W_Q = L_Q/\lambda = \frac{1}{\lambda}\frac{P_s\rho}{(1-\rho)^2} = \frac{P_s}{s\mu(1-\rho)^2}$.

上海交通大学 SHANGHAI JIAO TONG UNIVERSITY

- $L_Q = \sum_{n=s}^{\infty}(n-s)P_n = \sum_{n=s}^{\infty}(n-s)P_0\rho_n = \sum_{k=0}^{\infty} kP_0\rho_{s+k}$
  $= \sum_{k=1}^{\infty} kP_0\rho_s\rho^k = \sum_{k=1}^{\infty} kP_s\rho^k = \frac{P_s\rho}{(1-\rho)^2}$.
- Using Little's Law, $W_Q = L_Q/\lambda = \frac{1}{\lambda}\frac{P_s\rho}{(1-\rho)^2} = \frac{P_s}{s\mu(1-\rho)^2}$.
- $W = W_Q + \mathbb{E}[\text{service time}] = \frac{P_s}{s\mu(1-\rho)^2} + \frac{1}{\mu}$.

- $L_Q = \sum_{n=s}^{\infty}(n-s)P_n = \sum_{n=s}^{\infty}(n-s)P_0\rho_n = \sum_{k=0}^{\infty} kP_0\rho_{s+k}$
  $= \sum_{k=1}^{\infty} kP_0\rho_s\rho^k = \sum_{k=1}^{\infty} kP_s\rho^k = \frac{P_s\rho}{(1-\rho)^2}$.
- Using Little's Law, $W_Q = L_Q/\lambda = \frac{1}{\lambda}\frac{P_s\rho}{(1-\rho)^2} = \frac{P_s}{s\mu(1-\rho)^2}$.
- $W = W_Q + \mathbb{E}[\text{service time}] = \frac{P_s}{s\mu(1-\rho)^2} + \frac{1}{\mu}$.
- Using Little's Law,

- $L_Q = \sum_{n=s}^{\infty}(n-s)P_n = \sum_{n=s}^{\infty}(n-s)P_0\rho_n = \sum_{k=0}^{\infty} kP_0\rho_{s+k}$
  $= \sum_{k=1}^{\infty} kP_0\rho_s\rho^k = \sum_{k=1}^{\infty} kP_s\rho^k = \frac{P_s\rho}{(1-\rho)^2}.$
- Using Little's Law, $W_Q = L_Q/\lambda = \frac{1}{\lambda}\frac{P_s\rho}{(1-\rho)^2} = \frac{P_s}{s\mu(1-\rho)^2}.$
- $W = W_Q + \mathbb{E}[\text{service time}] = \frac{P_s}{s\mu(1-\rho)^2} + \frac{1}{\mu}.$
- Using Little's Law,
  $L = \lambda W = \lambda(W_Q + \frac{1}{\mu}) = L_Q + \frac{\lambda}{\mu} = \frac{P_s\rho}{(1-\rho)^2} + \frac{\lambda}{\mu}.$

- $L_Q = \sum_{n=s}^{\infty}(n-s)P_n = \sum_{n=s}^{\infty}(n-s)P_0\rho_n = \sum_{k=0}^{\infty} kP_0\rho_{s+k}$
  $= \sum_{k=1}^{\infty} kP_0\rho_s\rho^k = \sum_{k=1}^{\infty} kP_s\rho^k = \frac{P_s\rho}{(1-\rho)^2}$.
- Using Little's Law, $W_Q = L_Q/\lambda = \frac{1}{\lambda}\frac{P_s\rho}{(1-\rho)^2} = \frac{P_s}{s\mu(1-\rho)^2}$.
- $W = W_Q + \mathbb{E}[\text{service time}] = \frac{P_s}{s\mu(1-\rho)^2} + \frac{1}{\mu}$.
- Using Little's Law,
  $L = \lambda W = \lambda(W_Q + \frac{1}{\mu}) = L_Q + \frac{\lambda}{\mu} = \frac{P_s\rho}{(1-\rho)^2} + \frac{\lambda}{\mu}$.
- Due to unlimited capacity, arrival rate = entering rate, so the time average (W, W_Q) is based on all customers.

- $L_Q = \sum_{n=s}^{\infty}(n-s)P_n = \sum_{n=s}^{\infty}(n-s)P_0\rho_n = \sum_{k=0}^{\infty} kP_0\rho_{s+k}$
  $= \sum_{k=1}^{\infty} kP_0\rho_s\rho^k = \sum_{k=1}^{\infty} kP_s\rho^k = \frac{P_s\rho}{(1-\rho)^2}$.
- Using Little's Law, $W_Q = L_Q/\lambda = \frac{1}{\lambda}\frac{P_s\rho}{(1-\rho)^2} = \frac{P_s}{s\mu(1-\rho)^2}$.
- $W = W_Q + \mathbb{E}[\text{service time}] = \frac{P_s}{s\mu(1-\rho)^2} + \frac{1}{\mu}$.
- Using Little's Law,
  $L = \lambda W = \lambda(W_Q + \frac{1}{\mu}) = L_Q + \frac{\lambda}{\mu} = \frac{P_s\rho}{(1-\rho)^2} + \frac{\lambda}{\mu}$.
- Due to unlimited capacity, arrival rate = entering rate, so the time average (W, W_Q) is based on all customers.
- As $\rho \to 1$, all L, W, L_Q and W_Q tend to ∞.

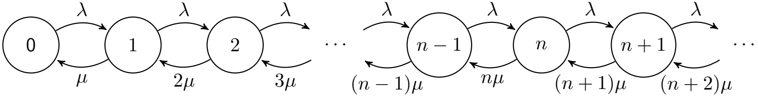
- By letting $s \to \infty$ we get the $M/M/\infty$ queue as a limiting case of the $M/M/s$ queue.

- By letting $s \to \infty$ we get the $M/M/\infty$ queue as a limiting case of the $M/M/s$ queue.
- Note: $M/M/\infty$ queue is always stable! (The *server utilization* is always 0.)

- By letting $s \to \infty$ we get the $M/M/\infty$ queue as a limiting case of the $M/M/s$ queue.
- Note: $M/M/\infty$ queue is always stable! (The *server utilization* is always 0.)
- All the measures can be obtained by letting $s \to \infty$ for those in the case of $M/M/s$ queue.†

† Use the Taylor series (泰勒级数): $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

- By letting $s \to \infty$ we get the $M/M/\infty$ queue as a limiting case of the $M/M/s$ queue.
- Note: $M/M/\infty$ queue is always stable! (The *server utilization* is always 0.)
- All the measures can be obtained by letting $s \to \infty$ for those in the case of $M/M/s$ queue.†
- Or, one can still derive P_n via the state space diagram:



† Use the Taylor series (泰勒级数): $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Theorem 6 (Limiting Distribution of $M/M/\infty$ Queue)

For an $M/M/\infty$ queue, its limiting (steady-state) distribution is given by

$$P_n = e^{-\lambda/\mu}\frac{(\lambda/\mu)^n}{n!}, \quad n \geq 0.$$

Theorem 6 (Limiting Distribution of $M/M/\infty$ Queue)

For an $M/M/\infty$ queue, its limiting (steady-state) distribution is given by

$$P_n = e^{-\lambda/\mu}\frac{(\lambda/\mu)^n}{n!}, \quad n \geq 0.$$

- In steady state, the number of customers in an $M/M/\infty$ station $\sim \text{Poisson}(\lambda/\mu)$.

Theorem 6 (Limiting Distribution of $M/M/\infty$ Queue)

For an $M/M/\infty$ queue, its limiting (steady-state) distribution is given by

$$P_n = e^{-\lambda/\mu}\frac{(\lambda/\mu)^n}{n!}, \quad n \geq 0.$$

- In steady state, the number of customers in an $M/M/\infty$ station $\sim \text{Poisson}(\lambda/\mu)$.
- Hence, $L = \sum_{n=0}^{\infty} nP_n = \mathbb{E}\left[\text{Poisson RV with mean } \frac{\lambda}{\mu}\right] = \frac{\lambda}{\mu}$.

Theorem 6 (Limiting Distribution of $M/M/\infty$ Queue)

For an $M/M/\infty$ queue, its limiting (steady-state) distribution is given by

$$P_n = e^{-\lambda/\mu}\frac{(\lambda/\mu)^n}{n!}, \quad n \geq 0.$$

- In steady state, the number of customers in an $M/M/\infty$ station $\sim \mathrm{Poisson}(\lambda/\mu)$.
- Hence, $L = \sum_{n=0}^{\infty} nP_n = \mathbb{E}\left[\text{Poisson RV with mean } \frac{\lambda}{\mu}\right] = \frac{\lambda}{\mu}$.
- Using Little's Law, $W = L/\lambda = \frac{1}{\mu}$.

Theorem 6 (Limiting Distribution of $M/M/\infty$ Queue)

For an $M/M/\infty$ queue, its limiting (steady-state) distribution is given by

$$P_n = e^{-\lambda/\mu}\frac{(\lambda/\mu)^n}{n!}, \quad n \geq 0.$$

- In steady state, the number of customers in an $M/M/\infty$ station $\sim \text{Poisson}(\lambda/\mu)$.
- Hence, $L = \sum_{n=0}^{\infty} nP_n = \mathbb{E}\left[\text{Poisson RV with mean } \frac{\lambda}{\mu}\right] = \frac{\lambda}{\mu}$.
- Using Little's Law, $W = L/\lambda = \frac{1}{\mu}$.
- $L_Q = 0$, $W_Q = 0$.

- $M/M/1/K$ Queue[†]
  - Customers arrive according to a Poisson process with rate λ.
  - The service times are iid random variables with $\mathrm{Exp}(\mu)$ distribution.
  - The customers are served in an FCFS fashion by a *single* server.
  - The capacity is K, $K \geq 1$, i.e., the maximal number of customers waiting in queue + customers in server $\leq K$.
  - A customer who finds the station is full (K customers there) leaves immediately (lost).
  - The entering rate, denoted as λ_e, is smaller than the arrival rate λ.
  - It is always stable (due to the finite capacity).

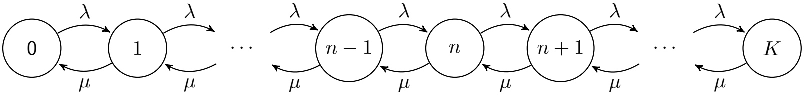
[†] $M/M/1/K$ Queue $\subset$ *Birth and Death Process with Finite Capacity* $\subset$ Continuous-Time Markov Chain.

- $M/M/1/K$ Queue†
    - Customers arrive according to a Poisson process with rate λ.
    - The service times are iid random variables with $\mathrm{Exp}(\mu)$ distribution.
    - The customers are served in an FCFS fashion by a *single* server.
    - The capacity is K, $K \geq 1$, i.e., the maximal number of customers waiting in queue + customers in server $\leq K$.
    - A customer who finds the station is full (K customers there) leaves immediately (lost).
    - The entering rate, denoted as λ_e, is smaller than the arrival rate λ.
    - It is always stable (due to the finite capacity).

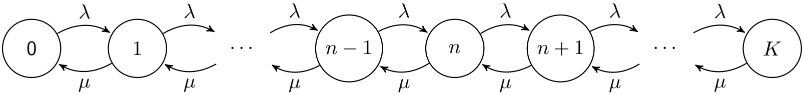
- In steady state
    - $\mathbb{P}(\text{station is full}) = P_K$.
    - Entering rate $\lambda_e = \lambda(1 - P_K)$.

† $M/M/1/K$ Queue $\subset$ *Birth and Death Process with Finite Capacity* $\subset$ Continuous-Time Markov Chain.

- The state space diagram is as follows:



- The state space diagram is as follows:

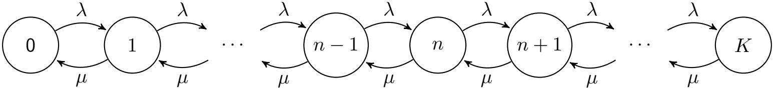


Theorem 7 (Limiting Distribution of $M/M/1/K$ Queue)

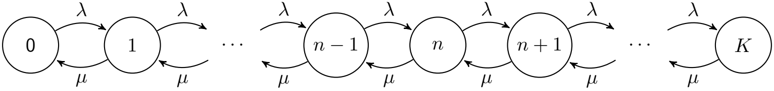
For an $M/M/1/K$ queue, its limiting (steady-state) distribution is given by

$$P_n = \begin{cases} \frac{(1-\rho)\rho^n}{1-\rho^{K+1}}, & \text{if } \rho \neq 1, \\ \frac{1}{K+1}, & \text{if } \rho = 1, \end{cases} \quad 0 \leq n \leq K,$$

where $\rho := \lambda/\mu$. (ρ is NOT the *server utilization*!)

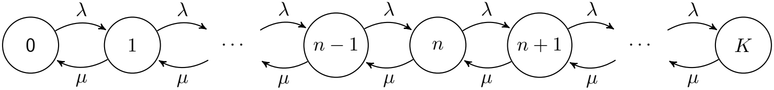


Proof.



Proof. Due to Observations 1 & 2,

State	Rate Process Leaves		Rate Process Enters
0	$P_0\lambda$	$=$	$P_1\mu$
n, $1 \le n \le K-1$	$P_n(\mu+\lambda)$	$=$	$P_{n-1}\lambda + P_{n+1}\mu$
K	$P_K\mu$	$=$	$P_{K-1}\lambda$



Proof. Due to Observations 1 & 2,

State	Rate Process Leaves		Rate Process Enters
0	$P_0\lambda$	$=$	$P_1\mu$
n, $1 \le n \le K-1$	$P_n(\mu+\lambda)$	$=$	$P_{n-1}\lambda + P_{n+1}\mu$
K	$P_K\mu$	$=$	$P_{K-1}\lambda$

Rewriting these equations gives

$$
\begin{aligned}
P_0\lambda &= P_1\mu, \\
P_n\lambda &= P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu), \quad 1 \le n \le K-1, \\
P_K\mu &= P_{K-1}\lambda.
\end{aligned}
$$

Or, equivalently,

$$
\begin{aligned}
P_0\lambda &= P_1\mu, \\
P_1\lambda &= P_2\mu + (P_0\lambda - P_1\mu) = P_2\mu, \\
P_2\lambda &= P_3\mu + (P_1\lambda - P_2\mu) = P_3\mu, \\
P_n\lambda &= P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu) = P_{n+1}\mu, \quad 1 \le n \le K-2, \\
P_{K-1}\lambda &= P_K\mu.
\end{aligned}
$$

Or, equivalently,

$$\begin{aligned} P_0\lambda &= P_1\mu, \\ P_1\lambda &= P_2\mu + (P_0\lambda - P_1\mu) = P_2\mu, \\ P_2\lambda &= P_3\mu + (P_1\lambda - P_2\mu) = P_3\mu, \\ P_n\lambda &= P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu) = P_{n+1}\mu, \quad 1 \le n \le K-2, \\ P_{K-1}\lambda &= P_K\mu. \end{aligned}$$

Let $\rho := \lambda/\mu$, solving in terms of P_0 yields

$$P_1 = P_0\rho,$$

Or, equivalently,

$$\begin{aligned}
P_0\lambda &= P_1\mu, \\
P_1\lambda &= P_2\mu + (P_0\lambda - P_1\mu) = P_2\mu, \\
P_2\lambda &= P_3\mu + (P_1\lambda - P_2\mu) = P_3\mu, \\
P_n\lambda &= P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu) = P_{n+1}\mu, \quad 1 \le n \le K-2, \\
P_{K-1}\lambda &= P_K\mu.
\end{aligned}$$

Let $\rho := \lambda/\mu$, solving in terms of P_0 yields

$$\begin{aligned}
P_1 &= P_0\rho, \\
P_2 &= P_1\rho = P_0\rho^2,
\end{aligned}$$

Or, equivalently,

$$
\begin{aligned}
P_0\lambda &= P_1\mu, \\
P_1\lambda &= P_2\mu + (P_0\lambda - P_1\mu) = P_2\mu, \\
P_2\lambda &= P_3\mu + (P_1\lambda - P_2\mu) = P_3\mu, \\
P_n\lambda &= P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu) = P_{n+1}\mu, \quad 1 \le n \le K-2, \\
P_{K-1}\lambda &= P_K\mu.
\end{aligned}
$$

Let $\rho := \lambda/\mu$, solving in terms of P_0 yields

$$
\begin{aligned}
P_1 &= P_0\rho, \\
P_2 &= P_1\rho = P_0\rho^2, \\
P_n &= P_{n-1}\rho = P_0\rho^n, \quad 1 \le n \le K.
\end{aligned}
$$

Or, equivalently,

$$
\begin{aligned}
P_0\lambda &= P_1\mu, \\
P_1\lambda &= P_2\mu + (P_0\lambda - P_1\mu) = P_2\mu, \\
P_2\lambda &= P_3\mu + (P_1\lambda - P_2\mu) = P_3\mu, \\
P_n\lambda &= P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu) = P_{n+1}\mu, \quad 1 \le n \le K-2, \\
P_{K-1}\lambda &= P_K\mu.
\end{aligned}
$$

Let $\rho := \lambda/\mu$, solving in terms of P_0 yields

$$
\begin{aligned}
P_1 &= P_0\rho, \\
P_2 &= P_1\rho = P_0\rho^2, \\
P_n &= P_{n-1}\rho = P_0\rho^n, \quad 1 \le n \le K.
\end{aligned}
$$

Since $1 = \Sigma_{n=0}^{K} P_n = P_0 \Sigma_{n=0}^{K} \rho^n = \begin{cases} P_0 \frac{1-\rho^{K+1}}{1-\rho}, & \text{if } \rho \neq 1, \\ P_0(K+1), & \text{if } \rho = 1, \end{cases}$ we have,

Or, equivalently,

$$
\begin{aligned}
P_0\lambda &= P_1\mu,\\
P_1\lambda &= P_2\mu + (P_0\lambda - P_1\mu) = P_2\mu,\\
P_2\lambda &= P_3\mu + (P_1\lambda - P_2\mu) = P_3\mu,\\
P_n\lambda &= P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu) = P_{n+1}\mu, \quad 1 \le n \le K-2,\\
P_{K-1}\lambda &= P_K\mu.
\end{aligned}
$$

Let $\rho := \lambda/\mu$, solving in terms of P_0 yields

$$
\begin{aligned}
P_1 &= P_0\rho,\\
P_2 &= P_1\rho = P_0\rho^2,\\
P_n &= P_{n-1}\rho = P_0\rho^n, \quad 1 \le n \le K.
\end{aligned}
$$

Since $1 = \Sigma_{n=0}^K P_n = P_0\Sigma_{n=0}^K \rho^n = \begin{cases} P_0\frac{1-\rho^{K+1}}{1-\rho}, & \text{if } \rho \neq 1,\\ P_0(K+1), & \text{if } \rho = 1,\end{cases}$ we have,

if $\rho \neq 1$, $\quad P_0 = \frac{1-\rho}{1-\rho^{K+1}}$, and $\quad P_n = \frac{(1-\rho)\rho^n}{1-\rho^{K+1}}$, $\quad 1 \le n \le K$;

Or, equivalently,

$$
\begin{aligned}
P_0\lambda &= P_1\mu,\\
P_1\lambda &= P_2\mu + (P_0\lambda - P_1\mu) = P_2\mu,\\
P_2\lambda &= P_3\mu + (P_1\lambda - P_2\mu) = P_3\mu,\\
P_n\lambda &= P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu) = P_{n+1}\mu, \quad 1 \le n \le K-2,\\
P_{K-1}\lambda &= P_K\mu.
\end{aligned}
$$

Let $\rho := \lambda/\mu$, solving in terms of P_0 yields

$$
\begin{aligned}
P_1 &= P_0\rho,\\
P_2 &= P_1\rho = P_0\rho^2,\\
P_n &= P_{n-1}\rho = P_0\rho^n, \quad 1 \le n \le K.
\end{aligned}
$$

Since $1 = \Sigma_{n=0}^{K} P_n = P_0 \Sigma_{n=0}^{K} \rho^n = \begin{cases} P_0 \frac{1-\rho^{K+1}}{1-\rho}, & \text{if } \rho \neq 1, \\ P_0(K+1), & \text{if } \rho = 1, \end{cases}$ we have,

if $\rho \neq 1$, $\quad P_0 = \frac{1-\rho}{1-\rho^{K+1}}$, and $\quad P_n = \frac{(1-\rho)\rho^n}{1-\rho^{K+1}}$, $\quad 1 \le n \le K$;

if $\rho = 1$, $\quad P_0 = \frac{1}{K+1}$, and $\quad P_n = \frac{1}{K+1}$, $\quad 1 \le n \le K$. ■

- If $\rho \neq 1$,
  $L = \sum_{n=0}^{K} nP_n$

- If $\rho \neq 1$,
  $L = \sum_{n=0}^{K} nP_n = \sum_{n=0}^{K} n\frac{(1-\rho)\rho^n}{1-\rho^{K+1}}$

- If $\rho \neq 1$,
  $L = \sum_{n=0}^{K} nP_n = \sum_{n=0}^{K} n\frac{(1-\rho)\rho^n}{1-\rho^{K+1}} = \frac{1-\rho}{1-\rho^{K+1}} \sum_{n=0}^{K} n\rho^n$

- If $\rho \neq 1$,
  $L = \sum_{n=0}^{K} nP_n = \sum_{n=0}^{K} n\frac{(1-\rho)\rho^n}{1-\rho^{K+1}} = \frac{1-\rho}{1-\rho^{K+1}}\sum_{n=0}^{K} n\rho^n$
  $= \frac{1-\rho}{1-\rho^{K+1}}\frac{\rho-(K+1)\rho^{K+1}+K\rho^{K+2}}{(1-\rho)^2}$

- If $\rho \neq 1$,

$$L = \sum_{n=0}^{K} nP_n = \sum_{n=0}^{K} n\frac{(1-\rho)\rho^n}{1-\rho^{K+1}} = \frac{1-\rho}{1-\rho^{K+1}} \sum_{n=0}^{K} n\rho^n$$
$$= \frac{1-\rho}{1-\rho^{K+1}} \frac{\rho-(K+1)\rho^{K+1}+K\rho^{K+2}}{(1-\rho)^2} = \frac{\rho}{1-\rho} \frac{1-(K+1)\rho^K+K\rho^{K+1}}{1-\rho^{K+1}}.$$

- If $\rho \neq 1$,
  $$L = \sum_{n=0}^{K} nP_n = \sum_{n=0}^{K} n\frac{(1-\rho)\rho^n}{1-\rho^{K+1}} = \frac{1-\rho}{1-\rho^{K+1}}\sum_{n=0}^{K} n\rho^n$$
  $$= \frac{1-\rho}{1-\rho^{K+1}}\frac{\rho-(K+1)\rho^{K+1}+K\rho^{K+2}}{(1-\rho)^2} = \frac{\rho}{1-\rho}\frac{1-(K+1)\rho^{K}+K\rho^{K+1}}{1-\rho^{K+1}}.$$
- If $\rho = 1$,
  $$L = \sum_{n=0}^{K} nP_n = \sum_{n=0}^{K} n\frac{1}{K+1} = \frac{1}{K+1}\frac{(K+1)K}{2} = \frac{K}{2}.$$

- If $\rho \neq 1$,
$$L = \sum_{n=0}^{K} nP_n = \sum_{n=0}^{K} n\frac{(1-\rho)\rho^n}{1-\rho^{K+1}} = \frac{1-\rho}{1-\rho^{K+1}}\sum_{n=0}^{K} n\rho^n$$
$$= \frac{1-\rho}{1-\rho^{K+1}}\frac{\rho-(K+1)\rho^{K+1}+K\rho^{K+2}}{(1-\rho)^2} = \frac{\rho}{1-\rho}\frac{1-(K+1)\rho^K+K\rho^{K+1}}{1-\rho^{K+1}}.$$

- If $\rho = 1$,
$$L = \sum_{n=0}^{K} nP_n = \sum_{n=0}^{K} n\frac{1}{K+1} = \frac{1}{K+1}\frac{(K+1)K}{2} = \frac{K}{2}.$$

- $\mathbb{P}[\text{station is full}] = P_K$.

- If $\rho \neq 1$,
  $$L = \sum_{n=0}^{K} nP_n = \sum_{n=0}^{K} n\frac{(1-\rho)\rho^n}{1-\rho^{K+1}} = \frac{1-\rho}{1-\rho^{K+1}} \sum_{n=0}^{K} n\rho^n$$
  $$= \frac{1-\rho}{1-\rho^{K+1}} \frac{\rho-(K+1)\rho^{K+1}+K\rho^{K+2}}{(1-\rho)^2} = \frac{\rho}{1-\rho} \frac{1-(K+1)\rho^{K}+K\rho^{K+1}}{1-\rho^{K+1}}.$$
- If $\rho = 1$,
  $$L = \sum_{n=0}^{K} nP_n = \sum_{n=0}^{K} n\frac{1}{K+1} = \frac{1}{K+1}\frac{(K+1)K}{2} = \frac{K}{2}.$$
- $\mathbb{P}[\text{station is full}] = P_K$.
- Entering rate $\lambda_e = \lambda(1 - P_K)$.

- If $\rho \neq 1$,
  $$L = \sum_{n=0}^{K} nP_n = \sum_{n=0}^{K} n\frac{(1-\rho)\rho^n}{1-\rho^{K+1}} = \frac{1-\rho}{1-\rho^{K+1}}\sum_{n=0}^{K} n\rho^n$$
  $$= \frac{1-\rho}{1-\rho^{K+1}}\frac{\rho-(K+1)\rho^{K+1}+K\rho^{K+2}}{(1-\rho)^2} = \frac{\rho}{1-\rho}\frac{1-(K+1)\rho^{K}+K\rho^{K+1}}{1-\rho^{K+1}}.$$
- If $\rho = 1$,
  $$L = \sum_{n=0}^{K} nP_n = \sum_{n=0}^{K} n\frac{1}{K+1} = \frac{1}{K+1}\frac{(K+1)K}{2} = \frac{K}{2}.$$
- $\mathbb{P}[\text{station is full}] = P_K$.
- Entering rate $\lambda_e = \lambda(1 - P_K)$.
- The *server utilization* $= \lambda_e/\mu = \rho(1 - P_K)$.

- If $\rho \neq 1$,
$$L = \sum_{n=0}^{K} nP_n = \sum_{n=0}^{K} n\frac{(1-\rho)\rho^n}{1-\rho^{K+1}} = \frac{1-\rho}{1-\rho^{K+1}}\sum_{n=0}^{K} n\rho^n$$
$$= \frac{1-\rho}{1-\rho^{K+1}}\frac{\rho-(K+1)\rho^{K+1}+K\rho^{K+2}}{(1-\rho)^2} = \frac{\rho}{1-\rho}\frac{1-(K+1)\rho^{K}+K\rho^{K+1}}{1-\rho^{K+1}}.$$

- If $\rho = 1$,
$$L = \sum_{n=0}^{K} nP_n = \sum_{n=0}^{K} n\frac{1}{K+1} = \frac{1}{K+1}\frac{(K+1)K}{2} = \frac{K}{2}.$$

- $\mathbb{P}[\text{station is full}] = P_K$.
- Entering rate $\lambda_e = \lambda(1-P_K)$.
- The *server utilization* $= \lambda_e/\mu = \rho(1-P_K)$.
- As $\rho \to \infty$, $L \to K$, $1-P_K \to 0$, $\rho(1-P_K) \to 1$.

- For those entered the station
  - The expected sojourn time $W = L/\lambda_e = \frac{L}{\lambda(1-P_K)}$.
  - The expected waiting time $W_Q = W - \frac{1}{\mu} = \frac{L}{\lambda(1-P_K)} - \frac{1}{\mu}$.

- For those entered the station
  - The expected sojourn time $W = L/\lambda_e = \frac{L}{\lambda(1-P_K)}$.
  - The expected waiting time $W_Q = W - \frac{1}{\mu} = \frac{L}{\lambda(1-P_K)} - \frac{1}{\mu}$.
- For ALL the arrivals (those who are lost have 0 sojourn time and waiting time)
  - The expected sojourn time $W' = (1 - P_K)W + 0 = \frac{L}{\lambda}$.
  - The expected waiting time $W'_Q = (1 - P_K)W_Q + 0 = \frac{L}{\lambda} - \frac{1-P_K}{\mu}$.

- For those entered the station
  - The expected sojourn time $W = L/\lambda_e = \frac{L}{\lambda(1-P_K)}$.
  - The expected waiting time $W_Q = W - \frac{1}{\mu} = \frac{L}{\lambda(1-P_K)} - \frac{1}{\mu}$.
- For ALL the arrivals (those who are lost have 0 sojourn time and waiting time)
  - The expected sojourn time $W' = (1-P_K)W + 0 = \frac{L}{\lambda}$.
  - The expected waiting time $W'_Q = (1-P_K)W_Q + 0 = \frac{L}{\lambda} - \frac{1-P_K}{\mu}$.
- The expected queue length $L_Q = \lambda_e W_Q = L - \rho(1-P_K)$,
  or, $= \lambda W'_Q = L - \rho(1-P_K)$.

- For those entered the station
  - The expected sojourn time $W = L/\lambda_e = \frac{L}{\lambda(1-P_K)}$.
  - The expected waiting time $W_Q = W - \frac{1}{\mu} = \frac{L}{\lambda(1-P_K)} - \frac{1}{\mu}$.
- For ALL the arrivals (those who are lost have 0 sojourn time and waiting time)
  - The expected sojourn time $W' = (1-P_K)W + 0 = \frac{L}{\lambda}$.
  - The expected waiting time $W'_Q = (1-P_K)W_Q + 0 = \frac{L}{\lambda} - \frac{1-P_K}{\mu}$.
- The expected queue length $L_Q = \lambda_e W_Q = L - \rho(1-P_K)$,
  or, $= \lambda W'_Q = L - \rho(1-P_K)$.
- As $\rho \to \infty$, $1-P_K \to 0$, $\rho(1-P_K) \to 1$, $L \to K$, $L_Q \to K-1$.

- For those entered the station
  - The expected sojourn time $W = L/\lambda_e = \frac{L}{\lambda(1-P_K)}$.
  - The expected waiting time $W_Q = W - \frac{1}{\mu} = \frac{L}{\lambda(1-P_K)} - \frac{1}{\mu}$.
- For ALL the arrivals (those who are lost have 0 sojourn time and waiting time)
  - The expected sojourn time $W' = (1-P_K)W + 0 = \frac{L}{\lambda}$.
  - The expected waiting time $W'_Q = (1-P_K)W_Q + 0 = \frac{L}{\lambda} - \frac{1-P_K}{\mu}$.
- The expected queue length $L_Q = \lambda_e W_Q = L - \rho(1-P_K)$,
  or, $= \lambda W'_Q = L - \rho(1-P_K)$.
- As $\rho \to \infty$, $1 - P_K \to 0$, $\rho(1-P_K) \to 1$, $L \to K$, $L_Q \to K-1$.
  - If μ is fixed and $\lambda \to \infty$:
    $\lambda(1-P_K) \to \mu$, $W \to \frac{K}{\mu}$, $W_Q \to \frac{K-1}{\mu}$, $W' \to 0$, $W'_Q \to 0$.

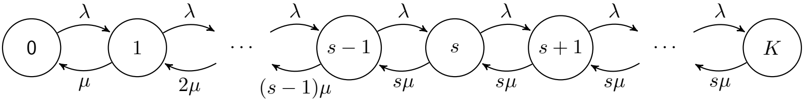
- For those entered the station
  - The expected sojourn time $W = L/\lambda_e = \frac{L}{\lambda(1-P_K)}$.
  - The expected waiting time $W_Q = W - \frac{1}{\mu} = \frac{L}{\lambda(1-P_K)} - \frac{1}{\mu}$.

- For ALL the arrivals (those who are lost have 0 sojourn time and waiting time)
  - The expected sojourn time $W' = (1-P_K)W + 0 = \frac{L}{\lambda}$.
  - The expected waiting time $W'_Q = (1-P_K)W_Q + 0 = \frac{L}{\lambda} - \frac{1-P_K}{\mu}$.

- The expected queue length $L_Q = \lambda_e W_Q = L - \rho(1-P_K)$,
  or, $= \lambda W'_Q = L - \rho(1-P_K)$.

- As $\rho \to \infty$, $1 - P_K \to 0$, $\rho(1-P_K) \to 1$, $L \to K$, $L_Q \to K-1$.
  - If μ is fixed and $\lambda \to \infty$:
    $\lambda(1-P_K) \to \mu$, $W \to \frac{K}{\mu}$, $W_Q \to \frac{K-1}{\mu}$, $W' \to 0$, $W'_Q \to 0$.
  - If λ is fixed and $\mu \to 0$:
    $\frac{1}{\mu}(1-P_K) \to \frac{1}{\lambda}$, $W \to \infty$, $W_Q \to \infty$, $W' \to \frac{K}{\lambda}$, $W'_Q \to \frac{K-1}{\lambda}$.

- $M/M/s/K$ queue† is a generalized version of $M/M/1/K$ queue. $(K \geq s)$
- The state space diagram is as follows:



- Let $s = 1$, it becomes the $M/M/1/K$ queue.
- Let $s = K$, it becomes the $M/M/K/K$ queue.
- There is no $M/M/\infty/K$ queue!

† $M/M/1/K$ Queue $\subset$ $M/M/s/K$ Queue $\subset$ *Birth and Death Process with Finite Capacity* $\subset$ CTMC.

Theorem 8 (Limiting Distribution of $M/M/s/K$ Queue)

For an $M/M/s/K$ queue, its limiting (steady-state) distribution is given by

$$P_n = \left[\sum_{i=0}^{s} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i + \varrho\right]^{-1} \rho_n \ , \quad 0 \le n \le K,$$

where $\rho := \lambda/(s\mu)$, (ρ is NOT the *server utilization*!) and

$$\varrho := \begin{cases} \frac{s^s}{s!}\frac{\rho^{s+1}(1-\rho^{K-s})}{1-\rho}, & \text{if } \rho \ne 1, \\ \frac{s^s}{s!}(K-s), & \text{if } \rho = 1, \end{cases}$$

and

$$\rho_n := \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n, & \text{if } 0 \le n \le s, \\ \frac{s^s}{s!}\rho^n, & \text{if } s+1 \le n \le K,\ K \ge s+1. \end{cases}$$

Theorem 8 (Limiting Distribution of $M/M/s/K$ Queue)

For an $M/M/s/K$ queue, its limiting (steady-state) distribution is given by

$$P_n = \left[\sum_{i=0}^{s} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i + \varrho\right]^{-1} \rho_n\ , \quad 0 \le n \le K,$$

where $\rho := \lambda/(s\mu)$, (ρ is NOT the *server utilization*!) and

$$\varrho := \begin{cases} \frac{s^s}{s!}\frac{\rho^{s+1}(1-\rho^{K-s})}{1-\rho}, & \text{if } \rho \neq 1, \\ \frac{s^s}{s!}(K-s), & \text{if } \rho = 1, \end{cases}$$

and

$$\rho_n := \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n, & \text{if } 0 \le n \le s, \\ \frac{s^s}{s!}\rho^n, & \text{if } s+1 \le n \le K,\ K \ge s+1. \end{cases}$$

- The *server utilization* $= \lambda_e/(s\mu) = \rho(1 - P_K)$.

- $M/G/1$ Queue[†]
  - Customers arrive according to a Poisson process with rate λ.
  - The service times are iid random variables with **arbitrary** distribution (mean: $\frac{1}{\mu}$, variance: σ^2).
  - The customers are served in an FCFS fashion by a *single* server.
  - The capacity is unlimited, i.e., waiting space is unlimited.
  - $M/G/1$ queue is stable **if and only if** $\lambda < \mu$.

[†] $M/G/1$ queue has an embedded discrete-time Markov chain.

- $M/G/1$ Queue[†]
  - Customers arrive according to a Poisson process with rate λ.
  - The service times are iid random variables with **arbitrary** distribution (mean: $\frac{1}{\mu}$, variance: σ^2).
  - The customers are served in an FCFS fashion by a *single* server.
  - The capacity is unlimited, i.e., waiting space is unlimited.
  - $M/G/1$ queue is stable **if and only if** $\lambda < \mu$.

- Let $m^2 := \left(\frac{1}{\mu}\right)^2 + \sigma^2$, and the *server utilization* $\rho := \lambda/\mu < 1$.
  - $\mathbb{P}(\text{the server is idle}) = 1 - \rho$.
  - $W_Q = \frac{\lambda m^2}{2(1-\rho)}$.
  - $L_Q = \lambda W_Q = \frac{\lambda^2 m^2}{2(1-\rho)}$.
  - $W = W_Q + \frac{1}{\mu} = \frac{\lambda m^2}{2(1-\rho)} + \frac{1}{\mu}$.
  - $L = \lambda W = L_Q + \lambda/\mu = \frac{\lambda^2 m^2}{2(1-\rho)} + \rho$.

[†] $M/G/1$ queue has an embedded discrete-time Markov chain.

上海交通大學 SHANGHAI JIAO TONG UNIVERSITY

- $M/G/1$ Queue†
  - Customers arrive according to a Poisson process with rate λ.
  - The service times are iid random variables with **arbitrary** distribution (mean: $\frac{1}{\mu}$, variance: σ^2).
  - The customers are served in an FCFS fashion by a *single* server.
  - The capacity is unlimited, i.e., waiting space is unlimited.
  - $M/G/1$ queue is stable **if and only if** $\lambda < \mu$.
- Let $m^2 := \left(\frac{1}{\mu}\right)^2 + \sigma^2$, and the *server utilization* $\rho := \lambda/\mu < 1$.
  - $\mathbb{P}(\text{the server is idle}) = 1 - \rho$.
  - $W_Q = \frac{\lambda m^2}{2(1-\rho)}$.
  - $L_Q = \lambda W_Q = \frac{\lambda^2 m^2}{2(1-\rho)}$.
  - $W = W_Q + \frac{1}{\mu} = \frac{\lambda m^2}{2(1-\rho)} + \frac{1}{\mu}$.
  - $L = \lambda W = L_Q + \lambda/\mu = \frac{\lambda^2 m^2}{2(1-\rho)} + \rho$.
- For $M/G/\infty$, the measures are the same as those in $M/M/\infty$.

† $M/G/1$ queue has an embedded discrete-time Markov chain.

Queueing Networks

- Queueing Network (multiple-station queueing system)
  - Customers can move from one station to another (for different service), before leaving the system.

- Queueing Network (multiple-station queueing system)
  - Customers can move from one station to another (for different service), before leaving the system.

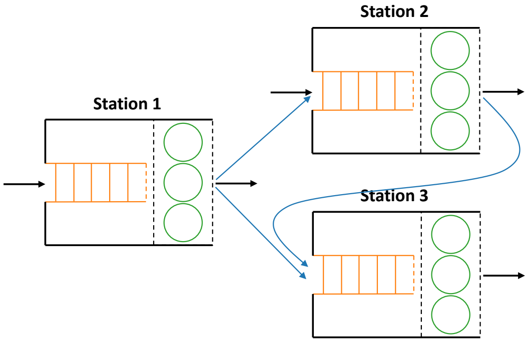


Figure: Illustration of Queueing Networks

- Jackson Queueing Network (first identified by Jackson (1963))†

  1. The network has J single-station queues.
  2. The jth station has s_j servers and a *single* queue.
  3. There is unlimited waiting space at each station (infinite capacity).
  4. Customers arrive at station j from outside according to a Poisson process with rate λ_j; all arrival processes are independent of each other.
  5. The service times at station j are iid random variables with $\mathrm{Exp}(\mu_j)$ distribution.

† Jackson network is an J-dimensional continuous-time Markov chain.

- Jackson Queueing Network (first identified by Jackson (1963))†

1. The network has J single-station queues.
2. The jth station has s_j servers and a *single* queue.
3. There is unlimited waiting space at each station (infinite capacity).
4. Customers arrive at station j from outside according to a Poisson process with rate λ_j; all arrival processes are independent of each other.
5. The service times at station j are iid random variables with $\mathrm{Exp}(\mu_j)$ distribution.
6. Customers finishing service at station i join the queue (if any) at station j with **routing probability** p_{ij}, or leave the network with probability p_{i0}, independently of each other.

† Jackson network is an J-dimensional continuous-time Markov chain.

- Jackson Queueing Network (first identified by Jackson (1963))[†]

  1. The network has J single-station queues.
  2. The jth station has s_j servers and a *single* queue.
  3. There is unlimited waiting space at each station (infinite capacity).
  4. Customers arrive at station j from outside according to a Poisson process with rate λ_j; all arrival processes are independent of each other.
  5. The service times at station j are iid random variables with $\mathrm{Exp}(\mu_j)$ distribution.
  6. Customers finishing service at station i join the queue (if any) at station j with **routing probability** p_{ij}, or leave the network with probability p_{i0}, independently of each other.
  7. A customer finishing service may be routed to the same station (i.e., re-enter).

[†] Jackson network is an J-dimensional continuous-time Markov chain.

- The routing probabilities p_{ij} can be put in a matrix form as follows:

$$\boldsymbol{P} := \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1J} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2J} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{3J} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{J1} & p_{J2} & p_{J3} & \cdots & p_{JJ} \end{bmatrix}.$$

- The matrix $\boldsymbol{P}$ is called the **routing matrix**.

- The routing probabilities p_{ij} can be put in a matrix form as follows:

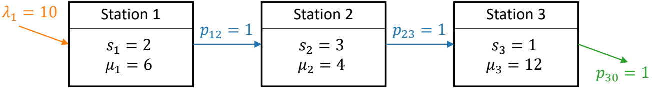
$$\boldsymbol{P} := \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1J} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2J} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{3J} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{J1} & p_{J2} & p_{J3} & \cdots & p_{JJ} \end{bmatrix}.$$

- The matrix $\boldsymbol{P}$ is called the **routing matrix**.

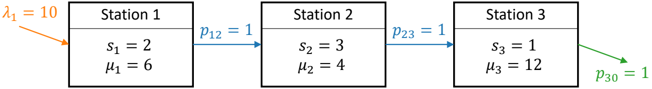
- Since a customer leaving station i either joints some other station, or leaves, we must have

$$\sum_{j=1}^{J} p_{ij} + p_{i0} = 1, \quad 1 \le i \le J.$$

- Example 1: Tandem Queue

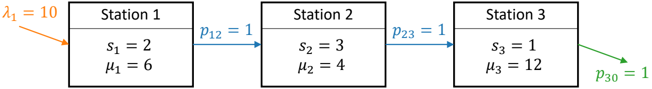


- Example 1: Tandem Queue



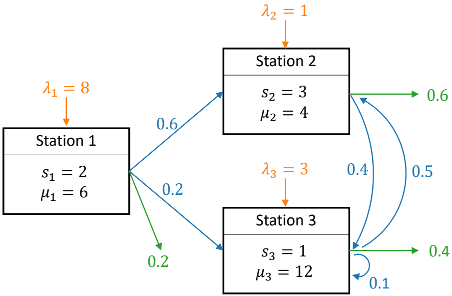
$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Example 1: Tandem Queue

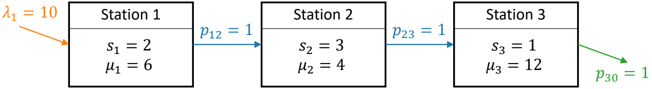


$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Example 2: General Network

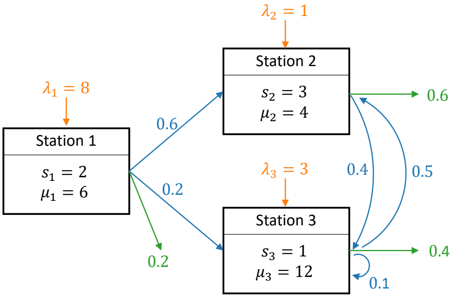


- Example 1: Tandem Queue



$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Example 2: General Network



$$\boldsymbol{P} = \begin{bmatrix} 0 & 0.6 & 0.2 \\ 0 & 0 & 0.4 \\ 0 & 0.5 & 0.1 \end{bmatrix}.$$

- Recall that customers arrive at station j from outside with rate λ_j.
- Let b_j be the rate of internal arrivals to station j.
- Then the total arrival rate to station j, denoted as a_j, is given by

$$a_j = \lambda_j + b_j, \quad 1 \le j \le J.$$

- Recall that customers arrive at station j from outside with rate λ_j.
- Let b_j be the rate of internal arrivals to station j.
- Then the total arrival rate to station j, denoted as a_j, is given by
$$a_j = \lambda_j + b_j, \quad 1 \leq j \leq J.$$
- If the stations are all **stable**
  - The departure rate of customers from station i will be the same as the total arrival rate to station i, namely, a_i.
  - The arrival rate of internal customers from station i to station j is $a_i p_{ij}$.

- Recall that customers arrive at station j from outside with rate λ_j.
- Let b_j be the rate of internal arrivals to station j.
- Then the total arrival rate to station j, denoted as a_j, is given by
$$a_j = \lambda_j + b_j, \quad 1 \leq j \leq J.$$
- If the stations are all **stable**
  - The departure rate of customers from station i will be the same as the total arrival rate to station i, namely, a_i.
  - The arrival rate of internal customers from station i to station j is $a_i p_{ij}$.
- Hence, $b_j = \sum_{i=1}^{J} a_i p_{ij}, \quad 1 \leq j \leq J.$

- Recall that customers arrive at station j from outside with rate λ_j.
- Let b_j be the rate of internal arrivals to station j.
- Then the total arrival rate to station j, denoted as a_j, is given by
$$a_j = \lambda_j + b_j, \quad 1 \le j \le J.$$
- If the stations are all **stable**
  - The departure rate of customers from station i will be the same as the total arrival rate to station i, namely, a_i.
  - The arrival rate of internal customers from station i to station j is $a_i p_{ij}$.
- Hence, $b_j = \sum_{i=1}^{J} a_i p_{ij}, \quad 1 \le j \le J.$
- Substituting in the pervious equation, we get the **traffic equations**:
$$a_j = \lambda_j + \sum_{i=1}^{J} a_i p_{ij}, \quad 1 \le j \le J.$$

- Let $\boldsymbol{a}^\mathsf{T} = [a_1\ a_2\ \cdots\ a_J]$ and $\boldsymbol{\lambda}^\mathsf{T} = [\lambda_1\ \lambda_2\ \cdots\ \lambda_J]$, the traffic equations can be written in matrix form as

$$\boldsymbol{a}^\mathsf{T} = \boldsymbol{\lambda}^\mathsf{T} + \boldsymbol{a}^\mathsf{T}\boldsymbol{P},$$

or

$$\boldsymbol{a}^\mathsf{T}(\boldsymbol{I} - \boldsymbol{P}) = \boldsymbol{\lambda}^\mathsf{T},$$

where $\boldsymbol{I}$ is the $J \times J$ identity matrix.

- Let $\boldsymbol{a}^\intercal = [a_1 \ a_2 \ \cdots \ a_J]$ and $\boldsymbol{\lambda}^\intercal = [\lambda_1 \ \lambda_2 \ \cdots \ \lambda_J]$, the traffic equations can be written in matrix form as

$$\boldsymbol{a}^\intercal = \boldsymbol{\lambda}^\intercal + \boldsymbol{a}^\intercal \boldsymbol{P},$$

or

$$\boldsymbol{a}^\intercal (\boldsymbol{I} - \boldsymbol{P}) = \boldsymbol{\lambda}^\intercal,$$

where $\boldsymbol{I}$ is the $J \times J$ identity matrix.

- Suppose the matrix $\boldsymbol{I} - \boldsymbol{P}$ is invertible, the above equation has a unique solution given by

$$\boldsymbol{a}^\intercal = \boldsymbol{\lambda}^\intercal (\boldsymbol{I} - \boldsymbol{P})^{-1}.$$

- Let $\boldsymbol{a}^\intercal = [a_1 \ a_2 \ \cdots \ a_J]$ and $\boldsymbol{\lambda}^\intercal = [\lambda_1 \ \lambda_2 \ \cdots \ \lambda_J]$, the traffic equations can be written in matrix form as
$$\boldsymbol{a}^\intercal = \boldsymbol{\lambda}^\intercal + \boldsymbol{a}^\intercal \boldsymbol{P},$$
or
$$\boldsymbol{a}^\intercal (\boldsymbol{I} - \boldsymbol{P}) = \boldsymbol{\lambda}^\intercal,$$
where $\boldsymbol{I}$ is the $J \times J$ identity matrix.

- Suppose the matrix $\boldsymbol{I} - \boldsymbol{P}$ is invertible, the above equation has a unique solution given by
$$\boldsymbol{a}^\intercal = \boldsymbol{\lambda}^\intercal (\boldsymbol{I} - \boldsymbol{P})^{-1}.$$

- The next theorem states the stability condition for Jackson networks in terms of the above solution.

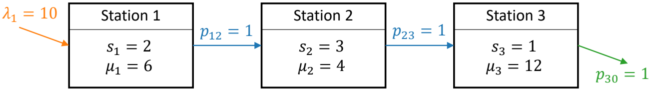
Theorem 9 (Stability of Jackson Networks)

A Jackson network with external arrival rate vector $\boldsymbol{\lambda}$ and routing matrix $\boldsymbol{P}$ is stable if:
(1) $\boldsymbol{I} - \boldsymbol{P}$ is invertible; and
(2) $a_i < s_i \mu_i$ for all $i = 1, 2, \ldots, J$, where a_i is given by the traffic equations.

Theorem 9 (Stability of Jackson Networks)

A Jackson network with external arrival rate vector $\boldsymbol{\lambda}$ and routing matrix $\boldsymbol{P}$ is stable if:
(1) $\boldsymbol{I} - \boldsymbol{P}$ is invertible; and
(2) $a_i < s_i\mu_i$ for all $i = 1, 2, \ldots, J$, where a_i is given by the traffic equations.

- Example 1: Tandem Queue

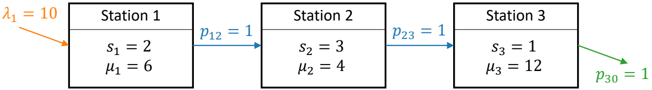


$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Theorem 9 (Stability of Jackson Networks)

A Jackson network with external arrival rate vector $\boldsymbol{\lambda}$ and routing matrix $\boldsymbol{P}$ is stable if:
(1) $\boldsymbol{I} - \boldsymbol{P}$ is invertible; and
(2) $a_i < s_i\mu_i$ for all $i = 1, 2, \ldots, J$, where a_i is given by the traffic equations.

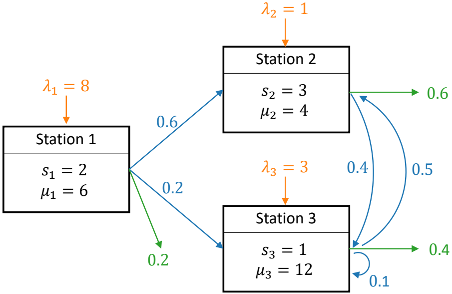
- Example 1: Tandem Queue



$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\lambda} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{a}^\intercal = \boldsymbol{\lambda}^\intercal(\boldsymbol{I} - \boldsymbol{P})^{-1} = [10\ 10\ 10].$$

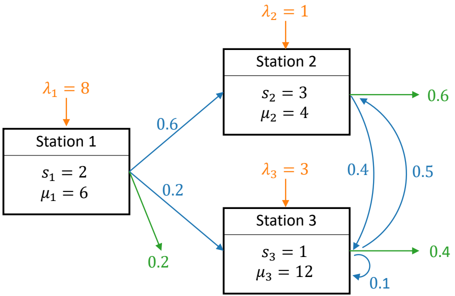
Stable.

- Example 2: General Network



$$P = \begin{bmatrix} 0 & 0.6 & 0.2 \\ 0 & 0 & 0.4 \\ 0 & 0.5 & 0.1 \end{bmatrix}.$$

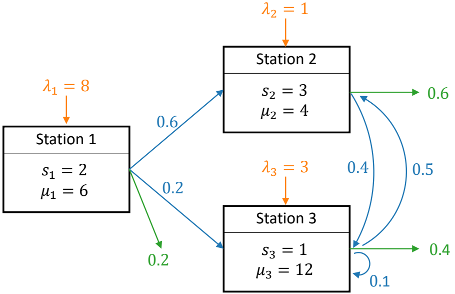
- Example 2: General Network



$$P = \begin{bmatrix} 0 & 0.6 & 0.2 \\ 0 & 0 & 0.4 \\ 0 & 0.5 & 0.1 \end{bmatrix}.$$

$$\boldsymbol{\lambda} = \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix}, \quad \boldsymbol{a}^\intercal = \boldsymbol{\lambda}^\intercal(\boldsymbol{I} - \boldsymbol{P})^{-1} = [8\ 10.7\ 9.9] \Rightarrow \text{Stable.}$$

- Example 2: General Network



$$\boldsymbol{P} = \begin{bmatrix} 0 & 0.6 & 0.2 \\ 0 & 0 & 0.4 \\ 0 & 0.5 & 0.1 \end{bmatrix}.$$

$$\boldsymbol{\lambda} = \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix}, \quad \boldsymbol{a}^\intercal = \boldsymbol{\lambda}^\intercal (\boldsymbol{I} - \boldsymbol{P})^{-1} = [8\ 10.7\ 9.9] \Rightarrow \text{Stable.}$$

If λ_2 is increased to 4,

$$\boldsymbol{\lambda} = \begin{bmatrix} 8 \\ 4 \\ 3 \end{bmatrix}, \quad \boldsymbol{a}^\intercal = \boldsymbol{\lambda}^\intercal (\boldsymbol{I} - \boldsymbol{P})^{-1} = [8\ 14.6\ 11.6] \Rightarrow \text{Unstable.}$$

- Let $L_j(t)$ be the number of customers in the jth station in a Jackson network at time t.

- Let $L_j(t)$ be the number of customers in the jth station in a Jackson network at time t.
- Then the state of the network at time t is given by $[L_1(t), L_2(t), \ldots, L_J(t)]$.

- Let $L_j(t)$ be the number of customers in the jth station in a Jackson network at time t.
- Then the state of the network at time t is given by $[L_1(t), L_2(t), \ldots, L_J(t)]$.
- When the Jackson network is stable, the limiting distribution of the sate of the network is

$$P(n_1, n_2, \ldots, n_J) = \lim_{t\to\infty} \mathbb{P}\{L_1(t) = n_1, L_2(t) = n_2, \ldots, L_J(t) = n_J\}.$$

- Let $L_j(t)$ be the number of customers in the jth station in a Jackson network at time t.
- Then the state of the network at time t is given by $[L_1(t), L_2(t), \dots, L_J(t)]$.
- When the Jackson network is stable, the limiting distribution of the sate of the network is

$$P(n_1, n_2, \dots, n_J) = \lim_{t \to \infty} \mathbb{P}\{L_1(t) = n_1, L_2(t) = n_2, \dots, L_J(t) = n_J\}.$$

- It is a joint probability.

Theorem 10 (Limiting Distribution of Jackson Network)

For a stable Jackson network, its limiting (steady-state) distribution is given by

$$P(n_1, n_2, \ldots, n_J) = P_1(n_1)P_2(n_2)\cdots P_J(n_J),$$

for $n_j = 0, 1, 2, \ldots$ and $j = 1, 2, \ldots, J$, where $P_j(n)$ is the limiting probability that there are n customers in an $M/M/s_j$ queue with arrival rate a_j and service rate μ_j.

Theorem 10 (Limiting Distribution of Jackson Network)

For a stable Jackson network, its limiting (steady-state) distribution is given by

$$P(n_1, n_2, \ldots, n_J) = P_1(n_1)P_2(n_2)\cdots P_J(n_J),$$

for $n_j = 0, 1, 2, \ldots$ and $j = 1, 2, \ldots, J$, where $P_j(n)$ is the limiting probability that there are n customers in an $M/M/s_j$ queue with arrival rate a_j and service rate μ_j.

- The limiting **joint** distribution of $[L_1(t), \ldots, L_J(t)]$ is a **product** of the limiting **marginal** distribution of $L_j(t)$, $j = 1, \ldots, J$.
$\Rightarrow$ Limiting behavior of all stations are independent of each other.

Theorem 10 (Limiting Distribution of Jackson Network)

For a stable Jackson network, its limiting (steady-state) distribution is given by

$$P(n_1, n_2, \ldots, n_J) = P_1(n_1)P_2(n_2)\cdots P_J(n_J),$$

for $n_j = 0, 1, 2, \ldots$ and $j = 1, 2, \ldots, J$, where $P_j(n)$ is the limiting probability that there are n customers in an $M/M/s_j$ queue with arrival rate a_j and service rate μ_j.

- The limiting **joint** distribution of $[L_1(t), \ldots, L_J(t)]$ is a **product** of the limiting **marginal** distribution of $L_j(t)$, $j = 1, \ldots, J$.
  $\Rightarrow$ Limiting behavior of all stations are independent of each other.
- The limiting distribution of station j is the same as that in an **isolated** $M/M/s_j$ queue with arrival rate a_j and service rate μ_j. (a_j's are solved from the **traffic equations**.)